ON CERTAIN GLOBAL CONSTRUCTIONS OF AUTOMORPHIC FORMS RELATED TO SMALL REPRESENTATIONS OF F_4

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To the memory of S. Rallis

Abstract. In this paper we consider some global constructions of liftings of automorphic representations attached to some commuting pairs in the exceptional group F_4. We consider two families of integrals. The first uses the minimal representation on the double cover of F_4, and in the second we consider examples of integrals of descent type associated with unipotent orbits of F_4.

1. **Introduction**

One of the important aspects of the Langlands conjectures is the study of correspondence of automorphic representations between two groups. Let H and G be two linear algebraic groups defined over a global field F. Given a homomorphism between the L groups of these two groups, the general conjectures predict a functorial lifting between automorphic representations of H and G.

There are several ways to study lifting of automorphic representations between two groups. Two powerful methods are the converse Theorem and the Arthur trace formula. The strength of these methods are their generality. On the other hand these methods are not explicit, in the sense that they do not actually construct the correspondence, but rather prove its existence.

A third method to construct these liftings is what we refer to as the small representations method. The idea of this method is as follows. Let M be a reductive group. Suppose that we can embed the groups G and H as a commuting pair inside M. By that we mean that we can embed these two groups inside M and under this embedding the two groups commute one with the other. Let Θ denote an automorphic representation of $M(\mathbf{A})$. Here $\mathbf{A}$ is the ring of adeles of a global field F. Let π denote an automorphic representation of $H(\mathbf{A})$. Then one can construct an automorphic function of $G(\mathbf{A})$ by means of the integral

$$f(g) = \int\limits_{H(F)\backslash H(\mathbf{A})} \int\limits_{V(F)\backslash V(\mathbf{A})} \varphi_\pi(h)\theta(v(h,g))\psi_V(v)dvdh \tag{1}$$

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Here V is a certain unipotent subgroup of M which is normalized by the embedding of G and H. Also ψ_V is a character of V which is preserved by these two groups. Finally, the function φ_π is a vector in the space of π, and θ is a vector in the space of Θ. Assuming that the above integral converges (this will happen, for example, if π is a cuspidal representation), denote by $\sigma(\pi, \Theta)$ the automorphic representation of $G(\mathbf{A})$ generated by all functions $f(g)$ defined above. The above discussion can be easily extended to automorphic representations of metaplectic covering groups of algebraic groups. Obviously, when considering integrals of the type of (1) defined over metaplectic covering groups, one should make sure that the cover splits. Otherwise the integrals will not be well defined.

Given the above construction, there are several natural problems regarding the relations between the representations π and $\sigma(\pi, \Theta)$. The first problem is the issue of the cuspidality of $\sigma(\pi, \Theta)$. In other words, what are the conditions on π and Θ, if any, so that $\sigma(\pi, \Theta)$ will be a cuspidal representation of $G(\mathbf{A})$. The second problem is to understand when $\sigma(\pi, \Theta)$ is nonzero. The third problem is to study the functoriality of the lift. Assume, for example, that $\sigma(\pi, \Theta)$ is a direct sum of irreducible automorphic representations. Then, one wants to check the relations between the unramified constituents of π and Θ with those of each irreducible summand of $\sigma(\pi, \Theta)$. There are other problems one can study. For example, when is the representation $\sigma(\pi, \Theta)$ irreducible. Another interesting problem is to try to characterize the image of the lift by means of a period integral. However, the above three problems are the basic ones, and should be studied prior to anything else. The machinery for studying these issues is quite routine. To verify cuspidality one needs to study the constant terms along unipotent radicals of maximal parabolic subgroups of G. The nonvanishing of the lift is usually done by showing that the function $f(g)$ has a certain nonzero Fourier coefficient. The unramified computation is done by the study of certain bilinear or trilinear forms.

We consider a few examples. There are two extreme cases. The first one is when the unipotent group is trivial. In this case integral (1) is given by

$$f(g) = \int\limits_{H(F)\backslash H(\mathbf{A})} \varphi_\pi(h)\theta((h, g))dh \tag{2}$$

The most well known example of this type is when M is the double cover of the symplectic group. In this example H is an orthogonal group and G itself is a symplectic group or its double cover. The representation Θ is the minimal representation which is defined on the double cover of $M(\mathbf{A})$. This case was studied by many authors. A reference for this example can be found in [R]. Other cases which involve the minimal representation can be found in [G-R-S4] where the group M is one of the exceptional groups of type E_6, E_7 and

E_8. We remark that there are constructions given by (2) which do not involve the minimal representation. Some cases in [G2] are such.

The other extreme case is when the group H is the trivial group. Thus, integral (1) is then given by

$$f(g) = \int\limits_{V(F)\backslash V(\mathbf{A})} \theta(vg)\psi_V(v)dv \tag{3}$$

In this case, which is known as the descent method, there is an automorphic representation π which is built inside the representation Θ. An example of this type can be found in [G-R-S2], [G-R-S3] and [G-R-S7].

Finally, there are also examples where both groups H and V are nonzero. See for example [G2], [G3] and [G4].

Prior to any computations it is natural to ask the question of how to construct lifting using integral (1). In other words, one would like to look for systematic ways to construct such examples. To give some heuristic of how to find such examples, it is convenient to use the language of unipotent orbits. In [G1], one associates with a unipotent orbit of a reductive group, a set of Fourier coefficients. This is done for the classical groups, however it is done in a similar way for the exceptional groups. In fact, in this paper, we work out this association in the case of the F split exceptional group F_4. Let σ denote an automorphic representation of a reductive group L. To this data we attach a set of unipotent orbits which we denote by $\mathcal{O}_L(\sigma)$. We say that $\mathcal{O} \in \mathcal{O}_L(\sigma)$ if σ has no nonzero Fourier coefficient associated with any unipotent orbit $\mathcal{O}'$ which is greater than $\mathcal{O}$. Also, the representation σ has a nonzero Fourier coefficient associated with the unipotent orbit $\mathcal{O}$. For more details on this set see [G1]. It is not known if $\mathcal{O}_L(\sigma)$ can contain more than one element. However, if it does contain only one element, this means that σ has no nonzero Fourier coefficient associated with any unipotent orbit which is greater than or not related to $\mathcal{O}_L(\sigma)$. Henceforth we shall assume that for all representations in question, this set consists of one element. We can then define the dimension of the representation σ to be $\dim\sigma = \frac{1}{2}\dim\mathcal{O}_L(\sigma)$. For basic properties of unipotent orbits and their dimensions, see [C-M].

To explain our method, let H and G be two reductive groups such that there is a homomorphism from ${}^LH$ to ${}^LG$. Let π denote an irreducible cuspidal representation of the group $H(\mathbf{A})$. Suppose that one can construct an automorphic representation Θ on a group $M(\mathbf{A})$, and assume that $\sigma(\pi, \Theta)$, as defined by integral (1), is a functorial lift from π corresponding to the above L group homomorphism. Then, in all known cases, the following dimension identity holds,

$$\dim\pi + \dim\Theta = \dim H + \dim V + \dim\sigma(\pi, \Theta) \tag{4}$$

It is important to emphasize that we do not claim that for any setup which satisfy equation (4), then integral (1) will give a functorial correspondence. In these notations we view the descent method as a limit case when H is the identity group, and hence its dimension is zero, and hence $\dim\pi = 0$.

To make things clear, we consider a few examples. Let $H = SO_{2n}$ be the split orthogonal group, and let $G = Sp_{2n}$. Then we have the L group homomorphism from $SO_{2n}(\mathbf{C})$ to $SO_{2n+1}(\mathbf{C})$. Let $M = \widetilde{Sp}_{4n^2}$, the double cover of the symplectic group. Let Θ denote the minimal representation of $M(\mathbf{A})$. Let π denote a generic irreducible cuspidal representation of $H(\mathbf{A})$. Then, it follows from [R], that integral (2) produces a functorial correspondence, and one can show that $\sigma(\pi, \Theta)$ is a generic representation as well. We verify identity (4) for this case. Indeed, in this case we have $\dim\pi = n^2 - n$, $\dim\Theta = 2n^2$, $\dim H = 2n^2 - n$, and $\dim\sigma(\pi, \Theta) = n^2$. The dimension of these representations are derived from the general formula for dimension of unipotent orbits as given in [C-M]. Thus, since π is generic, then $\mathcal{O}_{SO_{2n}}(\pi) = ((2n-1)1)$. The dimension of this orbit is $2(n^2 - n)$ and hence $\dim\pi = n^2 - n$. The representation Θ is associated with the minimal orbit which is (21^{4n^2-2}) and hence, it follows from [C-M] that its dimension is $2n^2$. It is now easy to verify identity (4) in this case.

As another example of this type, consider the case when $H = PGL_3$ and $G = G_2$. Here, Θ is the minimal representation of the exceptional group $E_6(\mathbf{A})$. It follows from [G-R-S4] that if π is an irreducible cuspidal representation of $PGL_3(\mathbf{A})$, and hence generic, then integral (2) produces a functorial lifting with $\sigma(\pi, \Theta)$ being generic. Since $\dim\pi = 3$, $\dim\Theta = 11$, $\dim PGL_3 = 8$ and $\dim\sigma(\pi, \Theta) = 6$, it follows that identity (4) holds.

As an another example we consider an example of a construction which is a descent construction, that is, uses the lifting as given by (3). Consider the case given in [G-R-S2] and [G-R-S3]. In this case one obtains the descent from cuspidal representations of $GL_{2n}(\mathbf{A})$ to cuspidal generic representations of $\widetilde{Sp}_{2n}(\mathbf{A})$. Even though the integral given for the descent in the above references involves also the theta representation of $\widetilde{Sp}_{2n}(\mathbf{A})$, it does not alter the identity (4). In the beginning of Section 4 we study in details these type of constructions. In the construction of the descent in this example, Θ is a certain residue of an Eisenstein series, and one can show (see [G-R-S7]) that this residue is attached to the unipotent orbit $((2n)^2)$ of Sp_{4n}. Thus $\dim\Theta = 4n^2 - n$. The dimension of V is $3n^2 - n$, and since $\sigma(\pi, \Theta)$ is generic it follows that $\dim\sigma(\pi, \Theta) = n^2$. Thus identity (4) holds. Strictly speaking this lift is not a functorial lift which corresponds to some L groups homomorphism. However, one can view it as an inverse map to the L group homomorphism from $Sp_{2n}(\mathbf{C})$ to $GL_{2n}(\mathbf{C})$.

In this paper we consider examples in the exceptional group F_4, of global constructions as given by integrals (2) and (3) which satisfy the dimension equation (4). More specifically,

in the notations of integrals (2) and (3), we will consider such integrals where $M = F_4$. Our main concern in this paper is to find conditions when such a construction produces a cuspidal image, and under what conditions the construction is nonzero. As follows from the beginning of Section three, in almost all global integrals of the type of integral (2), which satisfy the dimension equation (4), the representation Θ needs to be a minimal representation. In other words, we need $\mathcal{O}(\Theta) = A_1$. Section two is mainly devoted to the construction of such a representation on the double cover of F_4, and the study of its basic properties. This representation is defined as a certain residue of an Eisenstein series, essentially induced from the Borel subgroup. In addition, in that Section we also collect information about the structure of the Fourier coefficients of automorphic representations of $F_4(\mathbf{A})$ and its double cover.

In Section three we study integral (2) for five commuting pairs inside F_4. The pairs are $(SL_3 \times SL_3)$; $(SL_2 \times SL_2, Sp_4)$; (SL_2, SL_4); (SO_3, G_2) and (SL_2, Sp_6). In each case we study when the lifting from one to the other is cuspidal, and give a condition when it is nonzero. The computations are straightforward and use the properties of the minimal representation as were established in Section two.

In Section four we consider the descent map, that is integral (3) for some unipotent orbits of F_4. At subsection 4.1 we list all possible unipotent orbits of F_4, and using the dimension equation (81), which is a variant of the dimension equation (4), we obtain conditions on the the dimension of the automorphic representation involved in the construction. In subsection 4.2 we fix notations and some preliminary results concerning the nature of the answer we expect to get using the descent map. Finally, in subsection 4.3 we consider some examples in detail. That is, we study conditions for integral (3) to define a cuspidal representation, and conditions for the nonvanishing of the descent. The examples we choose to carry out are chosen mainly by our belief that they are of some interest.

As can be seen the missing ingredient in this paper is the local unramified theory. The main reason for this is that this issue is different in nature from the issue of cuspidality and the nonvanishing. Indeed, one of our goals in this paper is to show that when studying cuspidality and nonvanishing, the answer can be phrased in terms of the structure of the unipotent orbits of the group in question. In other words, when studying these two properties, the only ingredients we need to know about the automorphic representation Θ is what Fourier coefficients it supports. However, in subsection 3.6 we give a conjecture about the functorail lifting of each of the above five commuting pairs.

In Section five, we construct two examples of automorphic representations which are attached to specific unipotent orbits in F_4. As can be seen, unramified considerations do enter the calculations.

2. **The Minimal Representation of** F_4

2.1. **General Notations.** For $1 \le i \le 4$, let α_i denote the four simple roots of F_4. We label the roots of F_4 according to the diagram

$$\overset{\alpha_1}{0} - - - - \overset{\alpha_2}{0} ==>== \overset{\alpha_3}{0} - - - - \overset{\alpha_4}{0}$$

Here α_1, α_2 are the long simple roots and α_3, α_4 are the short simple roots.

Given a root, positive or negative, we denote by $\{x_\alpha(r)\}$ the one dimensional unipotent subgroup attached to the root α. For $1 \le i \le 4$, let $h_i(t_i)$ denote the one dimensional torus in F_4 which is associated to the SL_2 generated by $< x_{\pm\alpha_i}(r) >$. Then $h(t_1, t_2, t_3, t_4) = \prod_{i=1}^{4} h_i(t_i)$ is the maximal split torus of F_4. For $1 \le i \le 4$, we shall denote by $w[i]$ the simple reflection which corresponds to the simple root α_i. We shall write $w[i_1 i_2 \ldots i_m]$ for $w[i_1]w[i_2]\ldots w[i_m]$.

Let F be a global field, and let $\mathbf{A}$ be its ring of adeles. By ψ we denote a nontrivial character of $F\backslash\mathbf{A}$. We shall denote by J_n the matrix of order n which has ones on the other diagonal and zero elsewhere. The matrix $e_{i,j}$ will denote a matrix of order n which has one at the (i, j) entry, and zero elsewhere.

We denote by $\widetilde{F}_4$ the double cover of F_4. The construction of this group and its basic properties follows from [M].

Many of the computations done in this paper require the knowledge of commutating relations and conjugations which involves one parameter unipotent subgroups. We refer to [G-S] from which all the relevant data can be extracted.

Given an automorphic representation π and a unipotent subgroup V, we denote by φ_π^V its constant term along V. Here φ_π is a vector in the space of π. In other words, we denote

$$\varphi_\pi^V(g) = \int\limits_{V(F)\backslash V(\mathbf{A})} \varphi_\pi(vg)dv$$

In this paper we consider unipotent groups U and characters ψ_U which are defined on the group $U(F)\backslash U(\mathbf{A})$. Typically, these unipotent subgroups will be generated by one dimensional unipotent subgroups $x_\gamma(r)$ where γ is a positive root. For example, suppose that U is the one dimensional subgroup associated with the root γ. In this case we shall write $U = \{x_\gamma(r) : r \in R\}$ where R is a certain ring. When the ring R is clear we shall write $U = \{x_\gamma(r)\}$ for short. Given roots $\gamma_1, \ldots, \gamma_l$, positive or negative, we shall denote by

$< x_{\gamma_1}(r), \ldots, x_{\gamma_l}(r) >$ the group generated by all one dimensional unipotent subgroups $x_{\gamma_i}(r)$.

A convenient way to describe the character ψ_U is as follows. Let $\gamma_1, \ldots, \gamma_l$ denote l positive roots of F_4, and assume that the one dimensional unipotent subgroup $x_{\gamma_i}(r_i)$ are all in U but not in $[U, U]$. Given $u \in U$, write $u = x_{\gamma_1}(r_1) \ldots x_{\gamma_l}(r_l)u'$ where $u' \in U$ is any element which when written as a product of one dimensional unipotent subgroups associated with positive roots, then none of these roots are $\gamma_1, \ldots, \gamma_l$. Then define $\psi_U(u) = \psi_U(x_{\gamma_1}(r_1) \ldots x_{\gamma_l}(r_l)u') = \psi(a_1r_1 + \cdots + a_lr_l)$. Here $a_i \in F^*$.

2.2. **Unipotent Orbits and Fourier Coefficients in F_4.** In this subsection, let $G = F_4$. In this part we will describe how to associate to a given unipotent orbit in G, a set of Fourier coefficients. In [G1] it is explained how to construct this correspondence for automorphic representations of the classical groups. Another reference which studies unipotent orbits and Fourier coefficients for the group F_4 is [G-H].

According to the Bala-Carter classification, each unipotent orbit is represented by a diagram of G whose nodes are labelled by the numbers zero, one and two. We shall denote these numbers by ϵ_i for all $1 \leq i \leq 4$. A list of the possible diagrams can be found, for example, in [C] page 401. As usual an unlabelled node in the diagram corresponds to the number zero. Henceforth, we identify the set of unipotent orbits with the set of all such diagrams.

We associate to each diagram a set of Fourier coefficients. Let P be a parabolic subgroup of G. We list the parabolic subgroups of G according to the unipotent elements of the form $x_{\pm\alpha_i}(r)$ which are contained in the Levi part of the parabolic subgroup. Thus for example, we denote by P_{α_1} the parabolic subgroup whose Levi part is generated by $< x_{\pm\alpha_1}(r), T >$ where T is the maximal split torus of G. With these notations, the four maximal parabolic subgroups of G are $P_{\alpha_1,\alpha_2,\alpha_3}, P_{\alpha_1,\alpha_2,\alpha_4}, P_{\alpha_1,\alpha_3,\alpha_4}$ and $P_{\alpha_2,\alpha_3,\alpha_4}$. A similar notation will be used for the Levi part and the unipotent radical of a parabolic subgroup. For example, M_{α_1} and U_{α_1} will denote the Levi part and the unipotent radical of P_{α_1}.

To each unipotent orbit we attach a parabolic subgroup defined as follows. Suppose that $\Delta \subset \{\alpha_j : j \in \{1, 2, 3, 4\}\}$ is the set of simple roots in the diagram which are labeled zero. To this unipotent orbit we associate the parabolic subgroup P_Δ. We shall denote its Levi part by M_Δ, and its unipotent radical by U_Δ. For example, to the unipotent orbit, which is denoted by B_2, and whose diagram is given by

$$\overset{2}{0} - - - - 0 ==>== 0 - - - - \overset{1}{0}$$

we attach the parabolic subgroup P_{α_2,α_3}. Here $\epsilon_1 = 2,\ \epsilon_2 = \epsilon_3 = 0$ and $\epsilon_4 = 1$.

It will be convenient to confuse between a root α and the one parameter unipotent subgroup $\{x_\alpha(r)\}$. Thus, for example, if $\{x_\alpha(r)\} \subset U$ for some unipotent subgroup U, we will say that α is a root in U. By abuse of notations we will sometimes denote it by $\alpha \in U$. Given $\alpha = \sum_{i=1}^4 n_i\alpha_i$ we shall also denote this root by $(n_1n_2n_3n_4)$. Given a parabolic subgroup P_Δ as above, the set of roots in U_Δ are those roots $(n_1n_2n_3n_4)$ such that $\sum_{\alpha_i\notin\Delta} n_i > 0$. For example, the roots in U_{α_1,α_3} are the roots $(n_1n_2n_3n_4)$ such that $n_2 + n_4 > 0$. Once again, we emphasize that we are confusing a root with the one dimensional unipotent subgroup attached to this root.

Next, we determine a partition of all the roots in U_Δ. For any natural number n we denote $U'_\Delta(n) = \{\alpha \in U_\Delta : \sum_{i=1}^4 \epsilon_i n_i = n\}$. Let $U_\Delta(n)$ denote the unipotent subgroup of U_Δ which is generated by all one parameter subgroups $\{x_\alpha(r)\}$ such that $\alpha \in U'_\Delta(m)$ where $m \geq n$. Notice that $U_\Delta = U_\Delta(1)$ and if in the corresponding diagram all $\epsilon_i \neq 1$, then $U_\Delta = U_\Delta(2)$. We are mainly interested in the group $U_\Delta(2)$. It is not hard to check that M_Δ acts on this group.

As an example consider the above diagram attached to the unipotent orbit B_2. In this case, we have $\Delta = \{\alpha_2, \alpha_3\}$. The parabolic subgroup attached to it is P_{α_2,α_3}, and we can identify M_{α_2,α_3} with $GL_1^2 \times Sp_4$. We list the 20 roots in U_Δ according to the sets $U'_\Delta(n)$. We have

$$U'_\Delta(1) = \{(0001); (0011); (0111); (0121)\}$$

$$U'_\Delta(2) = \{(1000); (1100); (1110); (1120); (1220)\} \cup \{(0122)\}$$

$$U'_\Delta(3) = \{(1111); (1121); (1221); (1231)\}$$

$$U'_\Delta(4) = \{(1122); (1222); (1232); (1242); (1342)\} \qquad U'_\Delta(6) = \{(2342)\}$$

In general, we are interested in the action of M_Δ on the group $U_\Delta(2)/[U_\Delta(2), U_\Delta(1)]$. It follows from the general theory that M_Δ preserves this group and acts as a finite direct sum of irreducible representations. For example, for the unipotent orbit B_2, it follows from the above that M_Δ acts as a direct sum of a five dimensional irreducible representation and a one dimensional representation. We mention that this action of M_Δ can be lifted trivially to the unipotent group $U_\Delta(2)$.

Fix a unipotent orbit $\mathcal{O}$, and attach to it a set Δ as described above. Then, defined over the complex numbers $\mathbf{C}$, (or any other algebraically closed field), the group $M_\Delta(\mathbf{C})$, has an open orbit when acting on $U_\Delta(2)(\mathbf{C})$. Denote a representative of this orbit by $u_{\mathcal{O}}$. Thus, we may identify $u_{\mathcal{O}}$ with a unipotent element in $U_\Delta(2)(\mathbf{C})$. It follows from the general theory, see [C], that the connected component of the stabilizer of $u_{\mathcal{O}}$ inside $M_\Delta(\mathbf{C})$, is a reductive group. We shall denote this reductive group by $C(u_{\mathcal{O}})^0$. A list of these reductive groups is given in [C] page 401.

We now explain how to associate a set of Fourier coefficients to a unipotent orbit $\mathcal{O}$. Assume first that all nodes in the diagram associated with $\mathcal{O}$ are zeros or twos. Let Δ be as above and let $u_{\mathcal{O}}$ denote any unipotent element in $G(F)$ which lies in $U_\Delta(2)(F)$, such that the stabilizer of $u_{\mathcal{O}}$ inside $M_\Delta(F)$ is of the same type as $C(u_{\mathcal{O}})^0$. We consider a few examples. Suppose that $\mathcal{O}$ is the unipotent orbit labelled B_3. Its diagram is

$$\overset{2}{0} - - - - \overset{2}{0} ==>== 0 - - - - 0$$

Thus, $P_\Delta = P_{\alpha_3,\alpha_4}$ and $M_{\alpha_3,\alpha_4} = GL_1 \times GL_3$. We have

$$U'_\Delta(2) = \{(0100); (0110); (0111); (0120); (0121); (0122)\} \cup \{(1000)\}$$

Thus, the action of M_{α_3,α_4} on the group $U_\Delta(2)/[U_\Delta(2), U_\Delta(1)]$, and hence on the group $U_\Delta(2)$, is a sum of two irreducible representations. The first representation, is the six dimensional irreducible representation, which up to the action of the torus, is the symmetric square representation. The second representation is a one dimensional representation. According to [C] page 401, the group $C(u_\Delta)^0$ is of type A_1.

Thus, to define the corresponding Fourier coefficient, we look at all possible non-conjugate elements $u_0 \in U_\Delta(2)(F)$ such that the stabilizer inside $M_{\alpha_3,\alpha_4}(F)$, under its action on $U_\Delta(2)(F)$ as defined above, is a group of type A_1 defined over F. Since the action is via the symmetric square representation, one can choose the elements u_0 to be any element in the set

$$\{x_{1000}(1)x_{0100}(\beta_1)x_{0112}(\beta_2)x_{0122}(\beta_3) : \beta_j \in F^*\} \tag{5}$$

It is not hard to check that the stabilizer is an orthogonal group SO_3 which depend on the choice of β_j. Let φ be an automorphic function defined on $G(\mathbf{A})$. To a given element u_0 in the above set, we associate the Fourier coefficient

$$\int\limits_{U_\Delta(F)\backslash U_\Delta(\mathbf{A})} \varphi(u)\psi_{U,u_0}(u)du \tag{6}$$

where ψ_{U,u_0} is defined as follows. Write $u \in U_\Delta$ as $u = x_{1000}(r)x_{0100}(r_1)x_{0112}(r_2)x_{0122}(r_3)u'$ and define $\psi_{U,u_0}(u) = \psi(r + \beta_1 r_1 + \beta_2 r_2 + \beta_3 r_3)$. See subsection 2.1 for the precise notations. As we vary u_0 in the set of representatives given in (5), we associate with the unipotent orbit labelled B_3 a set of Fourier coefficients, given by integrals (6).

As an another example, consider the unipotent orbit labeled $F_4(a_1)$. Its diagram is

$$\overset{2}{0} - - - - \overset{2}{0} ==>== 0 - - - - \overset{2}{0}$$

Thus, the parabolic subgroup attached to this orbit is P_{α_3} and $M_{\alpha_3} = GL_1^2 \cdot GL_2$. In this case we have

$$U'_\Delta(2) = \{(0100); (0110); (0120)\} \cup \{(0001); (0011)\} \cup \{(1000)\}$$

It follows from [C] that the connected component of the stabilizer is the identity group. Consider the set of unipotent elements in $U_{\alpha_3}(F)$

$$\{x_{1000}(1)x_{0011}(1)x_{0100}(\beta_1)x_{0120}(\beta_2) : \beta_j \in F^*\}$$

It is not hard to check that the connected component of each such element, is the identity group. In a similar way as in (6), we associate with the unipotent orbit $F_4(a_1)$ a set of Fourier coefficients.

Next consider unipotent orbits where at least one of the nodes in the corresponding diagram is labelled with the number one. First assume that there is exactly one node which is labelled with one, and all other nodes are labeled with zero. There are exactly four such unipotent orbits which are associated with the four maximal parabolic subgroups. In this case we consider $U_\Delta(2)$ and proceed in a similar way as we did in the case where all nonzero nodes are labelled with twos. For example, consider the unipotent orbit $A_1 + \widetilde{A}_1$. The diagram attached to this orbit is

$$0 - - - - \overset{1}{0} ==>== 0 - - - - 0$$

Hence, the parabolic subgroup which corresponds to this orbit is $P_{\alpha_1,\alpha_3,\alpha_4}$. Its Levi part is $GL_2 \cdot SL_3$. From [C] we know that the connected component of the stabilizer is a group of type $A_1 \times A_1$. We have

$$U'_\Delta(2) = \{(1220); (1221); (1222); (1231); (1232); (1242)\}$$

The action of the Levi part on $U_\Delta(2)$ is as follows. The GL_2 part acts as a power of the determinant, and the SL_3 part via the symmetric square representation. As before, it is not hard to check that the set

$$\{x_{1220}(\beta_1)x_{1222}(\beta_2)x_{1242}(\beta_3) : \beta_j \in F^*\}$$

contains a set of representatives for all the orbits such that the connected component of the stabilizer inside $M_{\alpha_1,\alpha_3,\alpha_4}$ will be of type $A_1 \times A_1$. As in (6) we define

$$\int\limits_{U_\Delta(2)(F)\backslash U_\Delta(2)(\mathbf{A})} \varphi(u)\psi_{U_\Delta(2),u_0}(u)du \tag{7}$$

where $\psi_{U_\Delta(2),u_0}$ is defined as follows. Given $\beta_j \in F^*$, let $u_0 = x_{1220}(\beta_1)x_{1222}(\beta_2)x_{1242}(\beta_3)$. For $u \in U_\Delta(2)$ write $u = x_{1220}(r_1)x_{1222}(r_2)x_{1242}(r_3)u_1$ and define $\psi_{U_\Delta(2),u_0}(u) = \psi(\beta_1 r_1 + \beta_2 r_2 + \beta_3 r_3)$.

Finally, we need to consider the unipotent orbits whose corresponding diagram has one node labelled one and at least one more node which is labelled with a nonzero number. There are four such orbits. The way of attaching the Fourier coefficients to these orbits are similar to the way we did in the other cases. To make things clear, in each of the four cases we shall write the set of representatives of the various orbits under the action of $M_\Delta(F)$. Then, given an element u_0 in the corresponding set, we define the corresponding set of Fourier coefficients as in (7).

First consider the unipotent orbit B_2. Its diagram, the parabolic subgroup attached to this orbit, and the sets $U'_\Delta(n)$ were all described above. The connected component of the stabilizer is $A_1 \times A_1$. Consider the set

$$\{x_{1100}(1)x_{1120}(\beta_1)x_{0122}(\beta_2) : \beta_j \in F^*\}$$

Then it contains the set of all representatives of the various orbits under the action of $M_\Delta(F)$.

Next, we consider the unipotent orbit $\widetilde{A}_2 + A_1$. Its diagram is

$$0 - - - - \overset{1}{0} ==>== 0 - - - - \overset{1}{0}$$

The connected component of the stabilizer is a group of type A_1. We have

$$U'_\Delta(2) = \{(0111); (0121); (1111); (1121)\} \cup \{(1220)\}$$

The Levi part, which is $GL_2 \times GL_2$ acts on this set as the tensor product representation and as a one dimensional representation. In this case, $M_\Delta(F)$ acts transitively, and the representative of the open orbit is given by $x_{0121}(1)x_{1111}(1)x_{1220}(1)$.

The unipotent orbit labelled as $C_3(a_1)$ has the corresponding diagram

$$\overset{1}{0} - - - - 0 ==>== \overset{1}{0} - - - - 0$$

The connected component of the stabilizer is a group of type A_1. We have

$$U'_\Delta(2) = \{(0120); (0121); (0122)\} \cup \{(1110); (1111)\}$$

Hence, $M_\Delta = GL_2 \times GL_2$ acts as a sum of two irreducible representations. On the first representation, one copy of $GL_2(F)$ acts as the symmetric square representation and the other copy acts as a one dimensional representation. On the second irreducible representation one copy of $GL_2(F)$ acts as the standard representation and the other copy acts as a one dimensional representation. A set of unipotent representatives for the various orbits is included in the set

$$\{x_{0120}(\beta_1)x_{0122}(\beta_2)x_{1111}(1) : \beta_j \in F^*\}$$

The last case is the unipotent orbit labelled C_3. Its diagram is

$$\overset{1}{0} - - - - 0 ==>== \overset{1}{0} - - - - \overset{2}{0}$$

The connected component of the stabilizer is a group of type A_1. In this case the action is transitively, and as a representative of the open orbit, we can take the element $x_{0120}(1)x_{1110}(1)x_{0001}(1)$.

2.2.1. **On the Fourier Coefficients Attached to the Orbits $F_4(a_2)$ and $F_4(a_3)$.** For later reference we give some details concerning the Fourier coefficients of these two unipotent orbits. We start with $F_4(a_2)$. In this case $P_\Delta = P_{\alpha_1,\alpha_3}$. The roots in $U'_\Delta(2)$ are

$$(0001);\ (0011);\ (0100);\ (1100);\ (0110);\ (1110);\ (0120);\ (1120)$$

The group of characters defined on the group $U_\Delta(F)\backslash U_\Delta(\mathbf{A})$ is defined as follows. Write $u \in U_\Delta$ as $u = z(m_1, m_2)y(r_1, \ldots, r_6)u'$ where $u' \in [U_\Delta, U_\Delta]$, $z(m_1, m_2) = x_{0001}(m_1)x_{0011}(m_2)$ and

$$y(r_1, \ldots, r_6) = x_{0100}(r_1)x_{0110}(r_2)x_{0120}(r_3)x_{1100}(r_4)x_{1110}(r_5)x_{1120}(r_6)$$

Denote

$$\mathrm{Mat}'_{2\times 4} = \left\{ R \in \mathrm{Mat}_{2\times 4} : R = \begin{pmatrix} r_3 & r_4 & r_5 & r_6 \\ r_1 & r_2 & r_3 & -r_4 \end{pmatrix} \right\}$$

We mention that the motivation for dealing with this abelian group is from a certain matrix realization of the group $GSpin_7$. Embedding $GSpin_7$ inside GSO_8, the following described action is derived from the action of $M_\Delta = GL_2 \times GL_2$ on a unipotent radical of a maximal parabolic subgroup of $GSpin_7$.

Given a matrix A in

$$\mathrm{Mat}'_{4\times 2} = \left\{ A \in \mathrm{Mat}_{4\times 2} : A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_1 \\ a_6 & -a_3 \end{pmatrix} \right\}$$

and $\gamma_1, \gamma_2 \in F$ define for $u = z(m_1, m_2)y(r_1, \ldots, r_6)u' \in U_\Delta$ parameterized as above

$$\psi_{U_\Delta, A, \gamma_1, \gamma_2}(u) = \psi\left(\mathrm{tr}\left[A \begin{pmatrix} r_3 & r_4 & r_5 & r_6 \\ r_1 & r_2 & r_3 & -r_4 \end{pmatrix}\right]\right)\psi(\gamma_1 m_1 + \gamma_2 m_2)$$

The action of the Levi part of $M_{\alpha_1,\alpha_3}(F)$ on the group characters is given as follows.

First, let g be an element in $SL_2(F)$ which is generated by $< x_{\pm 1000}(r) >$. The action of this group is given by

$$\psi_{U_\Delta, A, \gamma_1, \gamma_2} \mapsto \psi_{U_\Delta, B, \gamma_1, \gamma_2} \qquad B = \begin{pmatrix} g & \\ & g^{-1} \end{pmatrix} A$$

Next, let $h \in SL_2(F)$ generated by $< x_{\pm 0010}(r) >$. Consider first the action of $x_{0010}(m)$. It is given by

$$\psi_{U_\Delta,A,\gamma_1,\gamma_2} \mapsto \psi_{U_\Delta,B,\gamma_1',\gamma_2'} \qquad B = \begin{pmatrix} 1 & & m & \\ & 1 & & -m \\ & & 1 & \\ & & & 1 \end{pmatrix} A \begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}; \qquad \begin{pmatrix} \gamma_1' \\ \gamma_2' \end{pmatrix} = \begin{pmatrix} 1 & \\ m & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$$

The action of $x_{-0010}(m)$ is defined similarly by taking the corresponding transpose matrices.

Finally, the action of the maximal torus $h(t_1, t_2, t_3, t_4)$ of F_4 is given by

$$\psi_{U_\Delta,A,\gamma_1,\gamma_2} \mapsto \psi_{U_\Delta,B,\gamma_1',\gamma_2'} \qquad B = T_1 A T_2; \quad \gamma_1' = t_3^{-1} t_4^2 \gamma_1; \quad \gamma_2' = t_2 t_3^{-1} t_4^{-1} \gamma_2$$

where $T_1 = \text{diag }(t_1^{-1} t_2 t_3^{-1} t_4, t_1 t_3^{-1} t_4, t_1^{-1} t_3, t_1 t_2^{-1} t_3)$ and $T_2 = \text{diag }(t_3 t_4^{-2}, t_2 t_3^{-1} t_4^{-1})$.

The Fourier coefficient (6) corresponds to the unipotent orbit $F_4(a_2)$ if and only if the connected component of the stabilizer of the character $\psi_{U_\Delta,A,\gamma_1,\gamma_2}$, is trivial.

The situation for $F_4(a_3)$ is similar. Here $P_\Delta = P_{\alpha_1,\alpha_3,\alpha_4}$ and $M_{\alpha_1,\alpha_3,\alpha_4}$ is generated by $SL_2 \times SL_3$ and the maximal split torus T. The roots in $U_\Delta'(2)$ are all 12 positive roots of the form $n_1\alpha_1 + \alpha_2 + n_3\alpha_3 + n_4\alpha_4$ where $n_i \geq 0$. There are 6 roots such that $n_1 = 0$ and 6 such that $n_1 = 1$. The six roots which have $n_1 = 0$ are $\{(0100); (0110); (0120); (0111); (0121); (0122)\}$. Write an element $u \in U_\Delta$ as $u = y(r_1, \ldots, r_6) z(m_1, \ldots, m_6) u'$ where

$$y(r_1, \ldots, r_6) = x_{0100}(r_1) x_{0110}(r_2) x_{0120}(r_3) x_{0111}(r_4) x_{0121}(r_5) x_{0122}(r_6) \tag{8}$$

and

$$z(m_1, \ldots, m_6) = x_{1100}(m_1) x_{1110}(m_2) x_{1120}(m_3) x_{1111}(m_4) x_{1121}(m_5) x_{1122}(m_6)$$

Here, $u' \in [U_\Delta, U_\Delta]$. We can relate these elements with the group $Mat^0_{3\times 3} = \{x \in Mat_{3\times 3} : J_3 x = x^t J_3\}$ where J_3 is the 3×3 matrix defined in subsection 2.1. The relation is given by

$$y(r_1, \ldots, r_6) \mapsto \begin{pmatrix} r_4 & r_5 & r_6 \\ r_2 & r_3 & r_5 \\ r_1 & r_2 & r_4 \end{pmatrix} \qquad z(m_1, \ldots, m_6) \mapsto \begin{pmatrix} m_4 & m_5 & m_6 \\ m_2 & m_3 & m_5 \\ m_1 & m_2 & m_4 \end{pmatrix}$$

To describe the characters of the group $U_\Delta(F)\backslash U_\Delta(\mathbf{A})$, let $A, B \in Mat^0_{3\times 3}$. Then define, for an element $u \in U_\Delta$ parameterized as above

$$\psi_{U_\Delta,A,B}(u) = \psi\left(\text{tr}\left[A \begin{pmatrix} r_4 & r_5 & r_6 \\ r_2 & r_3 & r_5 \\ r_1 & r_2 & r_4 \end{pmatrix} + B \begin{pmatrix} m_4 & m_5 & m_6 \\ m_2 & m_3 & m_5 \\ m_1 & m_2 & m_4 \end{pmatrix}\right]\right) \tag{9}$$

Thus we can identify the group characters of $U_\Delta(F)\backslash U_\Delta(\mathbf{A})$ by pairs (A, B) as above. The action of $M_{\alpha_1,\alpha_3,\alpha_4}(F)$ is as follows. First, given $g \in SL_3(F)$ we have $g(A, B) = (gAJ_3 g^t J_3, gBJ_3 g^t J_3)$. Then, for $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(F)$, we have $h(A, B) = (aA + bB, cA + dB)$. This action can be easily extended to an action of the group $GL_2 \times GL_3$. Doing so, we can describe the action of the torus T. We only need to describe the action of $h(1, t, 1, 1)$.

This is given by the above action using the diagonal matrix $g = \mathrm{diag}(t, 1, 1)$, and then $h = \mathrm{diag}(1, t^{-1})$.

The Fourier coefficient (6) attached to the character $\psi_{U_\Delta, A_0, B_0}$, corresponds to the unipotent orbit $F_4(a_3)$, if the connected component of the stabilizer of the pair (A_0, B_0) is trivial. This can be checked using the Lie algebras of these groups, and extending the above action to $GL_2 \times GL_3$. Thus, if $((h, g))(A_0, B_0) = (A_1, B_1)$ is an element in $GL_2 \times GL_3$, then differentiating, we obtain the two equations

$$g_1 A_0 + A_0 J_3 g_1^t J_3 + a_1 A_0 + b_1 B_0 = 0 \qquad g_1 B_0 + B_0 J_3 g_1^t J_3 + c_1 A_0 + d_1 B_0 = 0 \tag{10}$$

Here $g_1 \in Mat_{3\times 3}$ and $h_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ is a 2×2 matrix. Clearly, the group of matrices $(h_1, g_1) = (-2tI_2, tI_3)$ with $t \in F$, is a solution to these two equations. We refer to this solution as the trivial solution. Indeed, on the group level this solution corresponds to the torus element $(t^{-2} I_2, tI_3) \in GL_2 \times GL_3$, but from the above realization of the action on the unipotent matrices in F_4, this torus is not in $M_{\alpha_1, \alpha_3, \alpha_4}$.

Thus, if the solution to these two equations is only the trivial solution, then the Fourier coefficient (6) attached to the character $\psi_{U_\Delta, A_0, B_0}$ corresponds to the unipotent orbit $F_4(a_3)$.

2.2.2. **Root Exchange.** In the following Sections during the computations, we will carry out several Fourier expansions. One type of this expansions will repeat itself several times, and therefore it is convenient to state it in generality. We shall refer to this process as root exchange. This process was described in generality in [G-R-S7] subsection 7.1. This process has a local analogous which uses the notion of twisted Jacquet modules. In [G-R-S2] subsection 2.2, the global process stated in [G-R-S7] is formulated and carried out using the local language. In this paper, the proofs are global by nature, and therefore we prefer to use the global version. However, it should be emphasized that a similar proof can be stated and carried out in the local situation.

In this paper we will perform the expansions on a root by root process. For that reason we prefer to state the process of root exchange using a slightly different notations. We should also emphasize that the computations involved do not contribute any cocycle. This is true in both the global and the local version.

A typical integral that we start is an integral given by

$$\int\limits_{(F\backslash\mathbf{A})^2} \int\limits_{U(F)\backslash U(\mathbf{A})} f(ux_\alpha(m)x_\beta(r))\psi(m)\,dudmdr \tag{11}$$

Here f is an automorphic function, and α and β are two roots, need not be positive roots. Also, U is a certain unipotent group normalized by $x_\alpha(m)$ and $x_\beta(r)$. We assume that $[x_\beta(r), x_\alpha(m)] \in U$

Consider the following integral as a function of g,

$$L(g)=\int\limits_{F\backslash\mathbf{A}}\int\limits_{U(F)\backslash U(\mathbf{A})} f(ux_\alpha(m)g)\psi(m)dudm$$

and assume that it is left invariant under $x_\gamma(\delta)$ for all $\delta\in F$. That is $L(x_\gamma(\delta)g)=L(g)$. Here γ is any root, positive or negative, which satisfies the commutation relation $[x_\beta(r),x_\gamma(t)]=x_\alpha(c\beta t)u'$ with $u'\in U$. Here $c\in F^*$, a scalar which result from the structure constants in F_4. With these assumptions we can expand integral $L(g)$ along $x_\gamma(t)$ where $t\in F\backslash\mathbf{A}$. We obtain

$$\sum_{\delta\in F}\int\limits_{(F\backslash\mathbf{A})^2}\int\limits_{U(F)\backslash U(\mathbf{A})} f(ux_\alpha(m)x_\gamma(t)g)\psi(m+\delta t)dtdudmdr$$

From this we deduce that integral (11) is equal to

$$\int\limits_{F\backslash\mathbf{A}}\sum_{\delta\in F}\int\limits_{(F\backslash\mathbf{A})^2}\int\limits_{U(F)\backslash U(\mathbf{A})} f(ux_\alpha(m)x_\gamma(t)x_\beta(r))\psi(m+\delta t)dtdudmdr$$

Since f is automorphic then for all g and all $\delta\in F$ we have $f(x_\beta(\delta)g)=f(g)$. Using that, and the above commutation relations, the above integral is equal to

$$\int\limits_{F\backslash\mathbf{A}}\sum_{\delta\in F}\int\limits_{(F\backslash\mathbf{A})^2}\int\limits_{U(F)\backslash U(\mathbf{A})} f(ux_\alpha(m+\delta t)x_\gamma(t)x_\beta(r+\delta))\psi(m+\delta t)dtdudmdr$$

Changing variables, and collapsing summation over δ with integration over r, this integral is equal to

$$(12)\qquad \int\limits_{\mathbf{A}}\int\limits_{(F\backslash\mathbf{A})^2}\int\limits_{U(F)\backslash U(\mathbf{A})} f(ux_\alpha(m)x_\gamma(t)x_\beta(r))\psi(m)dtdudmdr$$

Arguing as in [Ga-S] one can easily show that the above integral is zero for all choice of data, if and only if the integral

$$(13)\qquad \int\limits_{(F\backslash\mathbf{A})^2}\int\limits_{U(F)\backslash U(\mathbf{A})} f(ux_\alpha(m)x_\gamma(t))\psi(m)dtdudm$$

is zero for all choice of data. Hence, we deduce that integral (11) is zero for all choice of data if and only if integral (12) or integral (13) are zero for all choice of data. Referring to this process we will say that we exchanged the root β by the root γ.

2.3. **Eisenstein series and their Residues.** In this Section we consider certain Eisenstein series on $G=F_4$ and study some of their residues. The basic reference for this type of construction is [K-P]. We also follow the ideas of the construction of a small representation of the double cover of odd orthogonal groups. This was done in [B-F-G1], and we refer to that paper for more details.

The Theta representation we construct will be a residue of an Eisenstein series associated with an induction from the Borel subgroup. We review how this is constructed. Let B denote the Borel subgroup of G, and let $T \subset B$ denote its maximal split torus. Let χ denote a character of T. Let $\widetilde{T}$ denote the inverse image of T inside $\widetilde{G}$. Let $Z(\widetilde{T})$ denote the center of $\widetilde{T}$, and let $\widetilde{T}_0$ denote any maximal abelian subgroup of $\widetilde{T}$. The character χ defines a genuine character of $Z(\widetilde{T})$ in the obvious way, and we extend it in any way to a character of $\widetilde{T}_0$. Inducing up to $\widetilde{T}$, extending it trivially to $\widetilde{B}$, and then inducing to $\widetilde{G}$, we obtain a representation of $\widetilde{G}$ which we denote by $Ind_{\widetilde{B}}^{\widetilde{G}}\chi$. It follows from [K-P] that this representation is uniquely determined by the character χ defined on $Z(\widetilde{T})$. These statements are true both locally and globally.

Let $\chi_{\bar{s}}$ denote the character of T defined as follows. Given $h(t_1, t_2, t_3, t_4) \in T$ we define $\chi_{\bar{s}}(h(t_1, t_2, t_3, t_4)) = |t_1|^{s_1}|t_2|^{s_2}|t_3|^{s_3}|t_4|^{s_4}$. Let $E_G^{(2)}(g, \bar{s})$ denote the Eisenstein series defined on $\widetilde{G}(\mathbf{A})$ which is associated with the induced representation $Ind_{\widetilde{B}(\mathbf{A})}^{\widetilde{G}(\mathbf{A})}\chi_{\bar{s}}\delta_B^{1/2}$. The poles of this Eisenstein series are determined by the intertwining operators corresponding to elements w of the Weyl group of G. The poles of these factors can be determined by using the factors

$$c_w(\chi_{\bar{s}}) = \prod_{\alpha>0, w(\alpha)<0} \frac{(1 - \chi_{\bar{s}}(a_\alpha)^{n(\alpha)})^{-1}}{(1 - q^{-1}\chi_{\bar{s}}(a_\alpha)^{n(\alpha)})^{-1}} \tag{14}$$

where $n(\alpha) = 1$ for the short roots and $n(\alpha) = 2$ for the long roots. Consider first the contribution from the long Weyl element in W. A simple application of (14) implies that the poles of the corresponding intertwining operator are determined by

$$Z_S(\bar{s}) = \frac{\zeta^S(2s_1)\zeta^S(2s_2)\zeta^S(s_3)\zeta^S(s_4)L^S(\bar{s})}{\zeta^S(2s_1+1)\zeta^S(2s_2+1)\zeta^S(s_3+1)\zeta^S(s_4+1)L^S(\bar{s}+1)}$$

Here the four partial zeta factors are the terms contributed from the simple roots α in the product in (14). The factor $L^S(\bar{s})$ is a product of 20 partial zeta factors evaluated at points of the form $\sum_{i=1}^4 n_i s_i$ with $n_i \geq 0$ and such that $n_1 + n_2 + n_3 + n_4 \geq 2$. The set S is a finite set, such that outside of S all places are finite unramified places. From this we deduce that $Z_S(\bar{s})$ has a simple multi pole at $s_1 = s_2 = \frac{1}{2}$ and $s_3 = s_4 = 1$. Its not hard to prove that all other intertwining operators are holomorphic at this point. Hence, the Eisenstein series $E_G^{(2)}(g, \bar{s})$ has a multi-residue at that point. Denote this multi-residue representation by $\Theta_G^{(2)}$. If there is no confusion we shall denote it simply by Θ. Thus, the representation Θ is a sub-quotient of the representation $Ind_{\widetilde{B}(\mathbf{A})}^{\widetilde{G}(\mathbf{A})}\chi_{\bar{s}_0}\delta_B^{1/2}$ where $\chi_{\bar{s}_0}(h(t_1, t_2, t_3, t_4) = |t_1t_2|^{1/2}|t_3t_4|$.

We will not need it, but we mention that the representation Θ is a subrepresentation of the induced representation $Ind_{\widetilde{B}(\mathbf{A})}^{\widetilde{G}(\mathbf{A})}\chi_\Theta$ where $\chi_\Theta(h(t_1, t_2, t_3, t_4)) = |t_1t_2|^{1/2}$.

Let $P = MU$ denote a maximal parabolic subgroup of $G = F_4$, where M is the Levi part of P, and U is its unipotent radical. Let M^0 denote the subgroup of M which is generated

by all copies of $SL_2 =< x_{\pm\alpha}(r) >$ where α is a positive root in M. There are four cases which we now list. First, if $P = P_{\alpha_1,\alpha_2,\alpha_3}$, then $M^0 = Spin_7$. When $P = P_{\alpha_1,\alpha_2,\alpha_4}$ or $P_{\alpha_1,\alpha_3,\alpha_4}$ then $M^0 = SL_2 \times SL_3$, and when $P = P_{\alpha_2,\alpha_3,\alpha_4}$ then $M^0 = Sp_6$.

Using induction by stages, we can write $Ind_{\widetilde{B}(\mathbf{A})}^{\widetilde{G}(\mathbf{A})}\chi_{\bar{s}_0}\delta_B^{1/2}$ as $Ind_{\widetilde{P}(\mathbf{A})}^{\widetilde{G}(\mathbf{A})}\tau_P\delta_P^{1/2}$, where τ_P is an automorphic representation of $\widetilde{M}(\mathbf{A})$. Thus, in the case when $P = P_{\alpha_1,\alpha_2,\alpha_3}$, then τ_P restricted to $\widetilde{M}^0(\mathbf{A}) = Spin_7^{(2)}(\mathbf{A})$, is a minimal representation of this group. Indeed, this follows by comparing the parameters between those of Θ and the parameters of the minimal representation of $Spin_7^{(2)}(\mathbf{A})$ as established in [B-F-G2]. In the case when $P = P_{\alpha_1,\alpha_2,\alpha_4}$ we obtain that τ_P restricted to $\widetilde{SL}_3(\mathbf{A}) \times SL_2(\mathbf{A})$ is the representation $\Theta_{SL_3} \times 1$, and similarly when $P = P_{\alpha_1,\alpha_3,\alpha_4}$ then we obtain the representation $\Theta_{SL_2} \times 1$ of $\widetilde{SL}_2(\mathbf{A}) \times SL_3(\mathbf{A})$. These two cases are obtained by comparing with the construction of the Theta representations as done in [K-P]. Finally, when $P = P_{\alpha_2,\alpha_3,\alpha_4}$ we obtain the right most residue representation of the Siegel Eisenstein series defined on $\widetilde{Sp}_6(\mathbf{A})$. This can be verified using the result of [I2]. Motivated by the above, let $\widetilde{M}_0$ denote the subgroup of $\widetilde{M}$ defined as follows. When M is the Levi part of $P_{\alpha_1,\alpha_2,\alpha_3}$ or of $P_{\alpha_2,\alpha_3,\alpha_4}$, we define $\widetilde{M}_0 = \widetilde{M}^0$. When M is the Levi part of $P_{\alpha_1,\alpha_2,\alpha_4}$, define $\widetilde{M}_0 = \widetilde{SL}_3 \times SL_2$, and in the last case, when M is the Levi part of $P_{\alpha_1,\alpha_3,\alpha_4}$, we define $\widetilde{M}_0 = \widetilde{SL}_2 \times SL_3$. A representation of the group $\widetilde{M}_0(\mathbf{A})$ will said to be a minimal representation if the only nontrivial Fourier coefficients this representation has, corresponds to the minimal orbit specified as follows. In the case when $\widetilde{M}_0 = Spin_7^{(2)}$ we refer to τ_P as a minimal representation if the only nonzero Fourier coefficients of this representation corresponds to the unipotent orbit (2^21^3). When $\widetilde{M}_0 = \widetilde{SL}_3(\mathbf{A}) \times SL_2(\mathbf{A})$ we refer to τ_P as a minimal representation if the only nonzero Fourier coefficients of this representation corresponds to the unipotent orbit (21) on $\widetilde{SL}_3$ and trivial on SL_2. For $\widetilde{M}_0 = \widetilde{SL}_2(\mathbf{A}) \times SL_3(\mathbf{A})$ we refer to τ_P as a minimal representation if it is trivial on SL_3. Finally when $\widetilde{M}_0 = \widetilde{Sp}_6$ we refer to τ_P as a minimal representation if the only nonzero Fourier coefficients of this representation corresponds to the unipotent orbit (21^4). It is a consequence of the above mentioned references that the representation τ_P restricted to $\widetilde{M}_0(\mathbf{A})$, is a minimal representation. The case where $M_0 = Sp_6$ follows from the Siegel-Weil identity as established in [I2].

Proposition 1. *Let $P = MU$ denote any one of the four maximal parabolic subgroup of $G = F_4$. With the above notations, the constant term $\Theta^U(g)$ when restricted to the group $\widetilde{M}_0(\mathbf{A})$ defines a minimal representation of this group. More over, the residue representation Θ is square integrable.*

Proof. We shall work out the details in the case $P = P_{\alpha_2,\alpha_3,\alpha_4}$. The other cases are done in a similar way.

Let $P = P_{\alpha_2,\alpha_3,\alpha_4}$. Then using induction by stages as above, we deduce that the representation Θ is a residue at $s = 27/32$ of the Eisenstein series $\widetilde{E}_{\tau_P}(g,s)$ associated with the induced representation $Ind_{\widetilde{P}(\mathbf{A})}^{\widetilde{G}(\mathbf{A})}\tau_P\delta_P^s$. To get this value of s, we start by noticing that $\chi_{\bar{s}_0}\delta_B^{1/2}(h(t_1,t_2,t_3,t_4)) = |t_1t_2|^{3/2}|t_3t_4|^2$. On Sp_6, we have the identity $|t_2|^{3/2}|t_3t_4|^2 = (\delta_{B(GL_3)}\delta_{P(Sp_6)}^{7/8})(h(1,t_2,t_3,t_4))$. Here $P(GL_3)$ is the maximal parabolic subgroup of Sp_6 whose Levi part is GL_3, and $B(GL_3)$ is the Borel subgroup of GL_3. Extending this character to T, we obtain

$$(\delta_{B(GL_3)}\delta_{P(Sp_6)}^{7/8})(h(t_1,t_2,t_3,t_4)) = |t_1|^{-\frac{21}{4}}|t_2|^{3/2}|t_3t_4|^2$$

We have $\delta_P^s(h(t_1,t_2,t_3,t_4)) = |t_1|^{8s}$. Hence, when matching the character $\delta_{B(GL_3)}\delta_{P(Sp_6)}^{7/8}\delta_P^s$ with $|t_1t_2|^{3/2}|t_3t_4|^2$ we get $s = 27/32$.

We need to study the constant term of this Eisenstein series. We use the method of [K-R]. See also [B-F-G2] and [G-R-S1] for similar cases. Consider the constant term along U. In other words, let

$$\widetilde{E}_{\tau_P}^U(g,s) = \int\limits_{U(F)\backslash U(\mathbf{A})} \widetilde{E}_{\tau_P}(ug,s)du$$

Unfolding the Eisenstein series for $\mathrm{Re}(s)$ large, we need to consider the space of double cosets $P(F)\backslash G(F)/P(F)$. This space has five elements, and as representatives, we can choose the five Weyl elements e, $w[1]$, $w[12321]$, $w[12324321]$ and the long Weyl element in this space which we denote by w_0. Notice that all of these elements are of order two, and hence $M_w = M_{w^{-1}}$.

We start with the contribution of w_0. Since Θ is a residue of this Eisenstein series, we deduce that at the point $s = 27/32$, where the residue occurs, the intertwining $(M_{w_0}f_s)(m)$ operator has a simple pole. Arguing as in [G-R-S1] pages 78-81 we deduce that at the bad places, after a suitable normalization by the local factors of the normalizing factor of the Eisenstein series, the local intertwining operators are holomorphic at the above point. Thus $(M_{w_0}f_s)(m)$ has a simple pole at $s = 27/32$. As a function of $g \in Sp_6^{(2)}(\mathbf{A})$, the function $(M_{w_0}f_s)(m)$ belongs to the space of τ_P restricted to the group $\widetilde{M}_0(\mathbf{A})$. As we stated before the Proposition this representation is a minimal representation.

Next we consider the contribution from the other four representatives. The term which corresponds to the identity is just the section which is clearly holomorphic. The three other representatives contributes each to the constant term an Eisenstein series defined on $\widetilde{Sp}_6(\mathbf{A})$. This Eisenstein series has the form $\widetilde{E}(m, M_wf_s, s')$ where M_wf_s is the corresponding intertwining operator and s' is a certain linear function in s. When $w = w[1]$ or when $w = w[12324321]$ we get the Eisenstein series associated to the induced representation $Ind_{Q(\mathbf{A})}^{\widetilde{Sp}_6(\mathbf{A})}\delta_Q^{s'}$ where Q is the maximal parabolic subgroup of Sp_6 whose Levi part is

GL_3. When $w = w[12321]$ we obtain an Eisenstein series associated with induction from the parabolic subgroup whose Levi part is $GL_1 \times Sp_4$.

This procedure is fairly standard. See [K-R], or [B-F-G2] for an example in the covering group. As an example, consider the case when $w = w[1]$. We have $w\alpha = \alpha$ when $\alpha = \pm(0010)$ and $\alpha = \pm(0001)$. Also $w(0100) = (1100)$. This means that w conjugates the subgroup $P(GL_3)$ into P. Here $P(GL_3)$ is the maximal parabolic subgroup of Sp_6 which contains the group GL_3. Thus, the contribution to $\widetilde{E}^U_{\tau_P}(g,s)$ from this Weyl element is

$$\sum_{\gamma \in P(GL_3)(F)\backslash Sp_6(F)} \int_{V(F)\backslash V(\mathbf{A})} \int_{U_w(\mathbf{A})} f_s(vwu\gamma g)dvdu$$

Here V denotes the unipotent radical of the parabolic subgroup $P(GL_3)$, and $U_w = < x_{1000}(r) >$. Thus, as a function of $m \in \widetilde{Sp}_6(\mathbf{A})$, this term is equal to $\widetilde{E}(m, M_w f_s, s')$, the Eisenstein series associated to the induced representation $Ind_{P(GL_3)\mathbf{A}}^{\widetilde{GSp}_6(\mathbf{A})} \delta_{P(GL_3)}^{s'}$. From the above integral we obtain that $s' = 2s + \frac{5}{16}$. It is also easy to verify that the intertwining operator $M_w f_s$ is holomorphic at $s = \frac{27}{32}$, and hence, we deduce that $\widetilde{E}(m, M_w f_s, s')$ is holomorphic at $s = \frac{27}{32}$ which corresponds to the point $s' = 2$.

The other two cases are similar, and in both we obtain that they are holomorphic at $s = 27/32$. Hence, all other four Weyl elements contributes a function to the constant term $\widetilde{E}^U_{\tau_P}(g,s)$, which is holomorphic at the point $s = 27/32$. From this the Proposition follows for this maximal parabolic subgroup P. As mentioned above, the other cases are similar and will be omitted.

Finally, to prove the square integrability we use Jacquet's criterion [J1]. This follows from the fact that $\Theta_G^{(2)}$ is a sub-representation of $Ind_{\widetilde{B}(\mathbf{A})}^{\widetilde{G}(\mathbf{A})} \chi_\Theta \delta_B^{1/2}$ where $\chi_\Theta(h(t_1,t_2,t_3,t_4)) = |t_1 t_2|^{-1/2} |t_3 t_4|^{-1}$ is in the negative Weyl chamber.

$\square$

Proposition 1 has a local version. Let Θ' denote any irreducible summand of Θ. Let ν denote any finite place where the local representation Θ'_ν is unramified. Then the representation Θ'_ν is the unramified subrepresentation of $Ind_{\widetilde{B}(F_\nu)}^{\widetilde{G}(F_\nu)} \chi_\Theta$. One can characterize this subrepresentation as the space of all functions $f \in Ind_{\widetilde{B}(F_\nu)}^{\widetilde{G}(F_\nu)} \chi_\Theta$ such that $I_w f = 0$ for all Weyl elements of F_4. Here I_w is the intertwining operator corresponding to w. This claim is a consequence of the periodicity Theorem in [K-P] adopted to the group F_4. It is all also simple to verify the claim that $I_w f = 0$ when w corresponds to a simple reflection. It should be mentioned that this intertwining operators need not converge at the point χ_Θ. In that case one views the above statement in the sense of meromorphic continuation.

Let $P = MU$ denote any one of the four maximal parabolic subgroups of G. Construct the Jacquet module $J_U(\Theta'_\nu)$. A representation of $\widetilde{M}_0(F_\nu)$ is said to be minimal, if it has no nonzero local functionals which corresponds to any unipotent orbit which is greater than the one specified in the global situation. As in the global case given in Proposition 1, we obtain

Corollary 1. *As a representation of $\widetilde{M}_0(F_\nu)$, the Jacquet module $J_U(\Theta'_\nu)$ is a minimal representation.*

Returning to the global case, to prove that Θ is indeed a minimal representation of the group $\widetilde{G}(\mathbf{A})$, we start by considering the Fourier coefficients which the Eisenstein series $\widetilde{E}_{\tau_P}(m, s)$ does not support. We will do it for the case when $P = P_{\alpha_2,\alpha_3,\alpha_4}$. To emphasize the relation of τ_P to the residue representation of $\widetilde{Sp}_6$, we shall write Θ_6 instead of τ_P. We also refer the reader to [C] page 440 for the description of the partial order of the unipotent orbits in F_4. We prove

Proposition 2. *Let $\mathcal{O}$ denote a unipotent orbit which is greater than or equal to the unipotent orbit $\widetilde{A}_2$. Then $\widetilde{E}_{\Theta_6}(m, s)$ has no nonzero Fourier coefficients corresponding to $\mathcal{O}$.*

Proof. The diagram which is attached to the unipotent orbit $\widetilde{A}_2$ is given by

$$0 - - - - 0 ==>== 0 - - - - \overset{2}{0}$$

In the notations of the previous subsection, we have

$$U'_\Delta(2) = \{(0001); (0011); (0111); (1111); (0121); (1121); (1221); (1231)\}$$

The character ψ_{U,u_Δ} can be defined as follows. Given $u \in U_\Delta$ write $u = x_{0121}(r_1)x_{1111}(r_2)u'$ and define $\psi_{U,u_\Delta}(u) = \psi(r_1 + r_2)$. To prove the Proposition, it is enough to prove that the integral

$$\int\limits_{U_\Delta(F)\backslash U_\Delta(\mathbf{A})} \widetilde{E}_{\Theta_6}(um, s)\psi_{U,u_\Delta}(u)du \tag{15}$$

is zero for all choice of data. It is also clear that it is enough to show this for $Re(s)$ large. In this proof, let $P = P_{\alpha_2,\alpha_3,\alpha_4}$ and $U = U_\Delta = U_{\alpha_1,\alpha_2,\alpha_3}$. Unfolding the Eisenstein series, we need to analyze the set $P(F)\backslash G(F)/U(F)$. It is clear that a set of representatives for this set can be chosen in the form wu_w where w is a Weyl element and u_w is a unipotent element inside $Spin_7(F)$. However, since the exceptional group G_2 is the stabilizer of the above character, it is in fact enough to consider representatives inside the set wu_w where wu_w is a representative of $P(F)\backslash G(F)/G_2(F)U(F)$. From this it is not hard to deduce that a set of representatives is contained inside the set

$$W_0 = \{e, w[123], w[1234], w[123243], w[123214323], w[1232143234], w[1232143213243]\}$$

This can be seen by first considering the set $P(F)\backslash G(F)/Spin_7(F)U(F)$ and then further study relevant double cosets of the form $R(F)\backslash Spin_7(F)/G_2(F)$ where R is a suitable maximal parabolic subgroup of $Spin_7$. We omit the details.

In other words we may choose representatives to be only Weyl elements. Thus we have

$$\int\limits_{U_\Delta(F)\backslash U_\Delta(\mathbf{A})} \widetilde{E}_{\Theta_6}(um,s)\psi_{U,u_\Delta}(u)du = \sum_{w\in W_0} \int\limits_{U_\Delta^w(F)\backslash U_\Delta(\mathbf{A})} f_s(wum)\psi_{U,u_\Delta}(u)du \tag{16}$$

Here $U_\Delta^w = w^{-1}U_\Delta w \in P$. We will now show that each summand of the right hand side is zero. If $w \in W_0$ is such that $wx_{1111}(r)w^{-1} \in U_{\alpha_2,\alpha_3,\alpha_4}$ then we get zero contribution form that summand, because f_s is left invariant under $U_{\alpha_2,\alpha_3,\alpha_4}(\mathbf{A})$ and ψ_{U,u_Δ} is not trivial on $x_{1111}(r)$. Since the Weyl elements $e, w[123], w[123243]$ and $w[123214323]$ have this property, they contribute zero.

As for the other three Weyl elements, we will use the minimality of Θ_6. See right before Proposition 1. Consider first the Weyl element $w[1234]$. It follows by direct conjugation that we obtain the integral

$$\int\limits_{(F\backslash\mathbf{A})^7} \theta_6\left(\begin{pmatrix} I_2 & X & Y \\ & I_2 & X^* \\ & & I_2 \end{pmatrix} g\right)\psi(\mathrm{tr}(X))dxdy$$

as an inner integration. Here $X \in Mat_{2\times 2}$, and Y and X^* are defined so that the above matrix is in Sp_6. This Fourier coefficient corresponds to the unipotent orbit (3^2) in Sp_6 (see [G1]), which is greater than the minimal orbit (21^4). Hence, by the minimality of Θ_6, it is zero for all choice of data.

Next consider the two Weyl elements $w[1232143234]$ and $w[1232143213243]$. In these two cases, we obtain the integral

$$\int\limits_{(F\backslash\mathbf{A})^5} \theta_6\left(\begin{pmatrix} 1 & x & y \\ & I_4 & x^* \\ & & 1 \end{pmatrix} g\right)\widetilde{\psi}(x)dxdy \tag{17}$$

or a conjugation of it by a Weyl element of Sp_6, as an inner integration. Here $x \in Mat_{1\times 4}$ and $y \in \mathbf{A}$. The character $\widetilde{\psi}$ is defined as follows. If $x = (x_{1,j}) \in Mat_{1\times 4}$, define $\widetilde{\psi}(x) = \psi(x_{1,1})$. To prove that this integral is zero we use the fact that Θ_6 is a minimal representation of $\widetilde{Sp}_6(\mathbf{A})$. Conjugate in the above integral by the discrete element

$$w' = \begin{pmatrix} 1 & & & \\ & & J_2 & \\ & -J_2 & & \\ & & & 1 \end{pmatrix}$$

where J_2 was defined in subsection 2.1. Then, expanding along the unipotent subgroup $x(r) = I_4 + re_{2,5}$, and using suitable conjugation, we obtain the integral

$$\int\limits_{(F\backslash \mathbf{A})^3} \theta_6\left(\begin{pmatrix} I_2 & & X \\ & I_2 & \\ & & I_2 \end{pmatrix}\right)\psi'(X)dX$$

Here $X = \begin{pmatrix} y & z \\ r & y \end{pmatrix}$ and $\psi'(X) = \psi(y)$. It is not hard to check that this Fourier coefficient corresponds to the unipotent orbit $(2^2 1^2)$ of Sp_6. See [G1]. By the minimality of Θ_6, this integral is zero for all choice of data. Thus integral (17) is zero for all choice of data.

Returning to the integral (16), we obtain that any summand on the right hand side is zero, and hence the integral on the left hand side of (16) is zero for $Re(s)$ large, and hence zero for all s. This proves that $\widetilde{E}_{\Theta_6}(m,s)$ has no nonzero Fourier coefficient with respect to the unipotent orbit $\widetilde{A}_2$.

Now we have to prove that for every unipotent orbit $\mathcal{O}$ which is greater than $\widetilde{A}_2$, the Eisenstein series has no nonzero Fourier coefficient which correspond to this orbit. This can be done in two ways. One way is to argue in a similar way as we did with the orbit $\widetilde{A}_2$. For example, it easy to prove this way that $\widetilde{E}_{\Theta_6}(m,s)$ is not generic, that is, it has no nonzero Fourier coefficient which correspond to the unipotent orbit whose label is F_4. Another way is to start with integral (15), use Fourier expansions and get the other orbits. For example, consider the orbit $\widetilde{A}_2 + A_1$. Its diagram is

$$0 - - - - \overset{1}{0} ==>== 0 - - - - \overset{1}{0}$$

and the corresponding Fourier coefficient was described in the previous subsection. Not to confuse with the group U_Δ as was defined in (15), for this proof only, we shall write V_Δ instead of U_Δ. Thus, we need to show that the integral

$$\int\limits_{V_\Delta(F)\backslash V_\Delta(\mathbf{A})} \widetilde{E}_{\Theta_6}(vm,s)\psi_{V,v_\Delta}(v)dv$$

is zero for all choice of data. Here ψ_{V,v_Δ} is defined as follows. For $v \in V_\Delta$, write $v = x_{0121}(r_1)x_{1111}(r_2)x_{1220}(r_3)v'$. Define $\psi_{V,v_\Delta}(v) = \psi(r_1 + r_2 + \beta r_3)$ where $\beta \in F^*$.

Let V denote the subgroup of V_Δ which consists of all roots in V_Δ such that the coefficient of α_4 is greater than zero. Thus $\dim V = 13$ and it is a subgroup of U_Δ as defined before integral (15). Notice that restricted to the group V we have $\psi_{V,v_\Delta} = \psi_{U,u_\Delta}$, where the right most character is defined in integral (15). Clearly it is enough to prove that the integral over $V(F)\backslash V(\mathbf{A})$ is zero. Starting with this Fourier coefficient, we expand along the unipotent

group $\{x_{0001}(l_1)x_{0011}(l_2)\}$ with points in $F\backslash\mathbf{A}$. We have

$$\int\limits_{V(F)\backslash V(\mathbf{A})} \widetilde{E}_{\Theta_6}(vm,s)\psi_{V,v_\Delta}(v)dv =$$

$$\int\limits_{V(F)\backslash V(\mathbf{A})} \sum_{\delta_i\in F} \int\limits_{(F\backslash\mathbf{A})^2} \widetilde{E}_{\Theta_6}(x_{0001}(l_1)x_{0011}(l_2)vm,s)\psi_{V,v_\Delta}(v)\psi(\delta_1 l_1+\delta_2 l_2)dl_i dv$$

Conjugating, from left to right, by the discrete elements $x_{0110}(-\delta_2)x_{1110}(-\delta_1)$ and changing variables, we obtain integral (15) as inner integration, which we proved to be zero for all choice of data. Thus, this Eisenstein series has no nonzero Fourier coefficients which corresponds to the unipotent orbit $\widetilde{A}_2+A_1$. Continuing similarly, we obtain the vanishing of all Fourier coefficients which corresponds to any unipotent orbit which is greater than $\widetilde{A}_2$. $\square$

2.4. **A Minimal Representation of F_4.** In this subsection we will prove that the residue of the Eisenstein series, constructed in the previous Sections and denoted there by Θ, is indeed a minimal representation for the double cover of F_4. In other words we will prove

Theorem 1. *Let $\mathcal{O}$ denote a unipotent orbit of F_4. Suppose that $\mathcal{O}$ is greater than the minimal orbit which is labeled by A_1. Then Θ has no nonzero Fourier coefficient which is attached to the unipotent orbit $\mathcal{O}$.*

Proof. We first explain the idea of the proof. Denote by $\mathcal{O}(\Theta)$ the set of all unipotent orbits of F_4 defined as follows. We have $\mathcal{O}\in\mathcal{O}(\Theta)$ if and only if the representation Θ has no nonzero Fourier coefficient associated with any unipotent orbit which is greater than or not related to the unipotent orbit $\mathcal{O}$. Also, we require that Θ do have a nonzero Fourier coefficient associated with the orbit $\mathcal{O}$. With these notations the statement of the Theorem is that $\mathcal{O}(\Theta)$ consists of one unipotent orbit which is the orbit A_1.

First, we prove that Θ has a nonzero Fourier coefficient corresponding to the unipotent orbit A_1. The diagram corresponding to this orbit is

$$\overset{1}{0}----0 ==>== 0----0$$

and the corresponding set of Fourier coefficients is given by

$$\int\limits_{F\backslash\mathbf{A}} \theta(x_{2342}(r)m)\psi(r)dr$$

It is clear that any nontrivial automorphic representation has such a nonzero Fourier coefficient. In particular it holds for the representation Θ.

From this and from Proposition 2 it follows that $\mathcal{O}(\Theta)$ consists of one unipotent orbit which is greater or equal than A_1, and which is less than or equal to the unipotent orbit

B_2. To prove the Theorem, we fix a unipotent orbit $\mathcal{O}$ which is greater than A_1 and less or equal to B_2. There are such five orbits. They are $B_2,\ A_2+\widetilde{A}_1,\ A_2,\ A_1+\widetilde{A}_1$, and $\widetilde{A}_1$. We will assume that $\mathcal{O}(\Theta)=\mathcal{O}$, where $\mathcal{O}$ is any one of these five orbits, and we shall derive a contradiction. The way to derive the contradiction is as follows. We consider the stabilizer of $\mathcal{O}$. It follows from [C] p. 401 that for all unipotent orbit $\mathcal{O}\neq A_2$, the stabilizer always contains a unipotent subgroup. This is also true for some Fourier coefficients associated with the unipotent orbit A_2, but not for all of them. We shall not need much information on the various unipotent orbit representatives of the orbit A_2. However, this information is contained in [I] Section 5. Assume that we are given a certain Fourier coefficient associated with the unipotent orbit $\mathcal{O}$. Suppose that it is given by the integral

$$\int\limits_{V(F)\backslash V(\mathbf{A})}\theta(vg)\psi_V(v)dv \tag{18}$$

and suppose also that the stabilizer of ψ_V contains an abelian unipotent subgroup Z. We then consider the Fourier coefficient

$$\int\limits_{Z(F)\backslash Z(\mathbf{A})}\ \int\limits_{V(F)\backslash V(\mathbf{A})}\theta(vz)\psi_V(v)\psi_Z(z)dzdv \tag{19}$$

Here ψ_Z is any character defined on $Z(F)\backslash Z(\mathbf{A})$. If we show that the above integral is zero for all choice of characters ψ_Z, this will prove that integral (18) is zero for all choice of data, and hence contradict the assumption that $\mathcal{O}(\Theta)=\mathcal{O}$. To show that the above integral is zero for all characters we use Fourier expansions to express the integral as a sum of two types of Fourier coefficients. The first type are Fourier coefficients which corresponds to unipotent orbits which are greater than or not related to $\mathcal{O}$. These coefficients will be zero by our assumption that $\mathcal{O}(\Theta)=\mathcal{O}$. The second type are Fourier coefficients of the type

$$\int\limits_{Y(F)\backslash Y(\mathbf{A})}\theta^{U(R)}(y)\psi_Y(y)dy$$

Here $\theta^{U(R)}$ is the constant term of the function θ along $U(R)$, where $U(R)$ is the unipotent radical of a maximal parabolic subgroup R of F_4. The group Y is a unipotent subgroup of $M(R)$, the Levi part of R. We then show that the character ψ_Y is a character which corresponds to a unipotent orbit of $M(R)$ which is not the minimal orbit. Then using Proposition 1 we deduce that this integral is zero.

We should mention that the proof is local by nature. Indeed, all the above ideas can be expressed by means of twisted Jacquet modules for a local constituent of an irreducible summand of the global representation Θ. We shall use this fact below. However, mainly

because of the Fourier expansions that we perform, it is convenient to use a global local argument.

We start with the unipotent orbit B_2. In other words, we shall assume that $\mathcal{O}(\Theta) = B_2$ and derive a contradiction. This unipotent orbit was described in subsection 2.2. A Fourier coefficient attached to this orbit is given by integral (18) where the roots in V are given in the beginning of subsection 2.2. The roots in V contains all 15 roots of the form $\alpha = (n_1n_2n_3n_4)$ with $n_1 \geq 1$, and the root (0122). Up to the action of $M(B_2) = T \cdot Sp_4$, a general character of the group V is defined as follows. Write $v = x_{1100}(r_1)x_{1120}(r_2)x_{0122}(r_3)v'$ where $v \in V$ and define $\psi_{V,\beta}(v) = \psi(r_1 + \beta r_2 + r_3)$. Here $\beta \in F^*$. From [C] we deduce that the stabilizer is a group of type $A_1 \times A_1$. In fact, when β is a square, then the stabilizer is the group $Spin_4 = SL_2 \times SL_2$ and when β is not a square we obtain the group $Spin(1,3)$ which depends on β. In both cases the stabilizer contains the unipotent subgroup generated by $\{x_{0100}(r_1)x_{0120}(-\beta r_1)\}$ and $\{x_{0110}(r_1)\}$. When β is a square, then after a suitable conjugation, we may choose $\psi_{V,\beta}$ as follows. Write as above $v = x_{1110}(r_1)x_{0122}(r_2)v'$ and define $\psi_{V,\beta}(v) = \psi(r_1 + r_2)$. We shall omit β from the notations and write ψ_V. With this choice the stabilizer contains the unipotent group $\{x_{0100}(m_1)x_{0120}(m_2)\}$. For simplicity we shall carry out the details when β is a square. The other case is similar.

We start by enlarging the group V to a group V_1 whose dimension is 18. To do so, consider the two roots (0111); (0121). Define the group V_1 to be the group generated by V and by $\{x_{0111}(r_1)x_{0121}(r_2)\}$. Then it follows from [G-R-S3] Lemma 1.1 that integral (18) is zero for all choice of data if and only if the integral

$$(20) \qquad \int\limits_{V_1(F)\backslash V_1(\mathbf{A})} \theta(v)\psi_V(v)dv$$

is zero for all choice of data. Here we view the character ψ_V as a character of V_1 by extending it trivially. This is well defined from the commutation relations in F_4. We also mention that the unipotent group $\{x_{0100}(m_1)x_{0120}(m_2)\}$ stabilizes the group V_1.

Choose Z to be the unipotent subgroup $\{x_{0120}(m_2)\}$. Our goal is to prove that integral (19), with V_1 replacing V, is zero for all characters of Z. In other words, we show that the integral

$$(21) \qquad \int\limits_{F\backslash \mathbf{A}} \int\limits_{V_1(F)\backslash V_1(\mathbf{A})} \theta(vx_{0120}(m))\psi_V(v)\psi(am)dmdv$$

is zero for all $a \in F$. Assume first that $a \neq 0$. In this case the above integral is a Fourier coefficient which corresponds to the unipotent orbit $C_3(a_1)$. Indeed, this Fourier coefficient was described in subsection 2.2. Using the left invariant properties of the function θ, we have $\theta(g) = \theta(w[4]g)$. Conjugating by $w[4]$ from left to right, we obtain exactly the Fourier

coefficient described in subsection 2.2. By our assumption on $\mathcal{O}(\Theta)$ this integral is zero. Next we consider the case when $a = 0$. We further expand along the unipotent group $\{x_{0100}(m_1)\}$. Consider first the contribution from the nontrivial orbit. Conjugating by $w[3]$ we obtain $w[3]x_{0100}(m_1)w[3]^{-1} = x_{0120}(m_1)$. Hence, when we consider the nontrivial character, we obtain integral (21), with a suitable $a \in F^*$, as inner integration. Hence we get zero.

We are left with the contribution of the trivial orbit. Therefore, it is enough to prove that the integral

$$\int\limits_{(F\backslash\mathbf{A})^2}\ \int\limits_{V_1(F)\backslash V_1(\mathbf{A})} \theta(vx_{0100}(m_1)x_{0120}(m_2))\psi_V(v)dm_1dm_2dv \tag{22}$$

is zero for all choice of data. Expand integral (22) along the unipotent abelian group $\{x_{0111}(r_1)x_{0121}(r_2)\}$. Thus, integral (22) is equal to

$$\sum_{\gamma_i\in F}\ \int\limits_{(F\backslash\mathbf{A})^2}\ \int\limits_{(F\backslash\mathbf{A})^2}\ \int\limits_{V_1(F)\backslash V_1(\mathbf{A})}$$

$$\theta(x_{0111}(r_1)x_{0121}(r_2)vx_{0100}(m_1)x_{0120}(m_2))\psi_V(v)\psi(\gamma_1r_1+\gamma_2r_2)dr_1dr_2dm_1dm_2dv$$

For all $\gamma_i \in F$ we have $\theta(g) = \theta(x_{0001}(-\gamma_2)x_{0011}(-\gamma_1)g)$. Plugging this into the above integral and changing variables, we obtain

$$\sum_{\gamma_i\in F}\ \int\limits_{(F\backslash\mathbf{A})^2}\ \int\limits_{(F\backslash\mathbf{A})^2}\ \int\limits_{V_1(F)\backslash V_1(\mathbf{A})}$$

$$\theta(x_{0111}(r_1)x_{0121}(r_2)vx_{0100}(m_1)x_{0120}(m_2)x_{0001}(-\gamma_2)x_{0011}(-\gamma_1))\psi_V(v)dr_1dr_2dm_1dm_2dv$$

Hence, to prove that integral (22) is zero for all choice of data, it is enough to prove that the integral

$$\int\limits_{(F\backslash\mathbf{A})^4}\ \int\limits_{V_1(F)\backslash V_1(\mathbf{A})} \theta(x_{0111}(r_1)x_{0121}(r_2)vx_{0100}(m_1)x_{0120}(m_2))\psi_V(v)dr_1dr_2dm_1dm_2dv \tag{23}$$

is zero for all choice of data.

Let V_2 denote the unipotent group generated by the group V_1 and the abelian group $\{x_{0111}(r_1)x_{0121}(r_2)x_{0100}(m_1)x_{0120}(m_2)\}$. Thus the dimension of V_2 is 20. Conjugating by the Weyl element $w[2134]$, integral (23) is equal to

$$\int \theta(x_{1000}(r_1)x_{0121}(r_2)v'x_{-1100}(m_1)x_{-1000}(m_2)w[2134])\psi(r_1+r_2)dr_1dr_2dm_1dm_2dv' \tag{24}$$

Here v' is a product over all other 16 one dimensional unipotent subgroups corresponding to roots in $w[2134]V_2w[2134]^{-1}$. All variables are integrated over $F\backslash\mathbf{A}$. We now apply Fourier

expansion to integral (24). Expand this integral along the unipotent subgroup $\{x_{1221}(t)\}$. Thus, integral (24) is equal to

$$\int \sum_{\gamma\in F} \int \theta(x_{1221}(t)x_{1000}(r_1)x_{0121}(r_2)v'x_{-1000}(m_1)\times \tag{25}$$

$$x_{-1100}(m_2)w[2134])\psi(r_1+r_2+\gamma t)dtdr_1dr_2dm_1dm_2dv'$$

We have

$$x_{-(1100)}(-\gamma)x_{1221}(t) = x_{0121}(-\gamma t)x_{1342}(t\gamma^2)x_{1221}(t)x_{-(1100)}(-\gamma)$$

The function θ is left invariant under $x_{-(1100)}(-\gamma)$. Performing the above conjugation in (25), changing variables and collapsing summation with integration, we obtain

$$\int_{\mathbf{A}} \int \theta(x_{1221}(t)x_{1000}(r_1)x_{0121}(r_2)v'x_{-1000}(m_1)\times \tag{26}$$

$$x_{-1100}(m_2)w[2134])\psi(r_1+r_2)dtdr_1dr_2dm_1dm_2dv'$$

where the adelic integration is over the variable m_2. This is the process of root exchange we refer to in subsection 2.2.2. Indeed, in the notations of that subsection, let $\alpha=(0121)$; $\beta=-(1100)$ and $\gamma=(1221)$. Thus we exchange the root $-(1100)$ by (1221). Next we repeat the same process, and we exchange the root $-(1000)$ by (1100). It follows that integral (26) is zero provided we can show that the integral

$$\int_{Y(F)\backslash Y(\mathbf{A})} \theta^{U(R),\psi}(y)\psi_Y(y)dy \tag{27}$$

is zero. Here, $R=P_{\alpha_2,\alpha_3,\alpha_4}$ is the maximal parabolic subgroup of F_4 whose Levi part is GSp_6, and $U(R)$ is its unipotent radical. Also,

$$\theta^{U(R),\psi}(g) = \int_{U(R)(F)\backslash U(R)(\mathbf{A})} \theta(ug)\psi_{U(R)}(u)du$$

where $\psi_{U(R)}$ is defined as follows. Write $u\in U(R)$ as $u=x_{1000}(r)u'$. Then $\psi_{U(R)}(u)=\psi_{U(R)}(x_{1000}(r)u')=\psi(r)$. Finally, the group Y consists of all roots $\{(0010);(0011);(0120);(0121);(0122)\}$. The character ψ_Y is defined by $\psi_Y(y)=\psi_Y(x_{0121}(m_1)y')=\psi(m_1)$. We now do two more exchange of roots. First we exchange the root (0110) by (0011), and then exchange (0111) by (0010). Then, conjugating by the Weyl element $w[43]$, integral (27) is zero for all choice of data if and only if the integral

$$\int_{Y_1(F)\backslash Y_1(\mathbf{A})} \theta^{U(R),\psi}(y_1w[43])\psi_{Y_1}(y_1)dy_1 \tag{28}$$

is zero for all choice of data. Here Y_1 is the unipotent subgroup which consists of the roots $\{(0110);(0111);(0120);(0121);(0122)\}$, and $\psi_{Y_1}(y_1)=\psi_{Y_1}(x_{0110}(r)y_1')=\psi(r)$.

Next, we expand along the group $x_{0100}(t)$. Thus, integral (28) is a sum of integrals of the form

$$\int_{F\backslash\mathbf{A}}\int_{Y_1(F)\backslash Y_1(\mathbf{A})}\theta^{U(R),\psi}(x_{0100}(r)y_1)\psi_{Y_1}(y_1)\psi(\gamma r)drdy_1 \tag{29}$$

where $\gamma\in F$.

Conjugating by the element $x_{0010}(-\gamma)$, and changing variables we obtain that the integral (29) is zero provided the integral

$$\int_{F\backslash\mathbf{A}}\int_{Y_1(F)\backslash Y_1(\mathbf{A})}\theta^{U(R),\psi}(x_{0100}(r)y_1)\psi_{Y_1}(y_1)drdy_1 \tag{30}$$

is zero for all choice of data. Thus integral (27) is zero for all choice of data if integral (30) is zero for all choice of data. Expand integral (30) along the unipotent group $x_{0001}(m_1)x_{0011}(m_2)$. The contribution from the nontrivial orbit is zero. Indeed, in this case we obtain

$$\int_{Y_1(F)\backslash Y_1(\mathbf{A})}\int_{(F\backslash\mathbf{A})^2}\theta^{U(R),\psi}(x_{0001}(m_1)x_{0011}(m_2)y_1)\psi'_{Y_1}(y_1)\psi(\gamma_1m_1+\gamma_2m_2)dm_1dm_2dy_1 \tag{31}$$

where $\gamma_1,\gamma_2\in F$ are not both zero. As follows from subsection 2.2 this Fourier coefficient is associated with the unipotent orbit $F_4(a_1)$, and hence zero for all choice of data. Thus we are left with integral (31) where $\gamma_1=\gamma_2=0$. In this case we can write integral (31) as

$$\int_{Y_2(F)\backslash Y_2(\mathbf{A})}\theta^{V(L)}(y_2)\psi_{Y_2}(y_2)dy_2$$

Here $L=P_{\alpha_1,\alpha_2,\alpha_3}$ is the maximal parabolic subgroup of F_4 whose Levi part is $GSpin_7$. We denote its unipotent radical by $V(L)$, and $\theta^{V(L)}$ is the constant term along $V(L)$. The group Y_2 is a unipotent subgroup of $GSpin_7$. It consists of all positive roots in that group except (0010). Thus its dimension is eight. The character ψ_{Y_2} is defined as follows $\psi_{Y_2}(y_2)=\psi_{Y_2}(x_{1000}(t_1)x_{0110}(t_2)y_2')=\psi(t_1+t_2)$. This Fourier coefficient is associated with the unipotent orbit (51^2) of $Spin_7$. Applying Proposition 1 this integral is zero. This completes the case of the unipotent orbit B_2, when β as defined before integral (21) is a square. As mentioned above, the case when β is not a square is similar and will be omitted.

Next we assume that $\mathcal{O}(\Theta)=A_2+\widetilde{A}_1$. The corresponding Fourier coefficient was not described explicitly, and we do it now. In this case the set $U'_\Delta(2)$ consists of all nine roots of the form $\sum n_i\alpha_i$ where $n_3=2$. Thus dim $U_\Delta(2)=14$ and write $V=U_\Delta(2)$. Then the corresponding Fourier coefficient is given by integral (18) where ψ_V is defined as follows. Write $v=x_{1220}(r_1)x_{0122}(r_2)x_{1121}(r_3)v'$. Then $\psi_V(v)=\psi(r_1+r_2+r_3)$. As stated in [C], the stabilizer of this character is a group of type A_1, and it can be identified with the split

orthogonal group SO_3. Hence it contains a unipotent subgroup. This unipotent subgroup is generated by $\{x_{1000}(r)x_{0100}(-r)x_{1100}(ar^2)x_{0001}(r)\}$ where $a \in F^*$.

For the unipotent orbit A_2 the situation is different. In this case the group $V = U_{\alpha_2,\alpha_3,\alpha_4}$, and the stabilizer of this orbit is a group of type A_2. It follows from [I] that over the rational points there is a choice of a character ψ_V such that the stabilizer is the group $SL_3(F)$. But there is also a choice of characters such that the stabilizer is various unitary groups. The character ψ_V whose stabilizer is SL_3, is given as follows. Let $v = x_{1000}(r_1)x_{1342}(r_2)v'$. Then $\psi_V(v) = \psi(r_1 + r_2)$. A unipotent subgroup which is contained in the stabilizer is, for example, $\{x_{0010}(m_1)x_{0001}(m_2)x_{0011}(m_3)\}$. We shall refer to this Fourier coefficient as to the split Fourier coefficient associated with the unipotent orbit A_2.

To prove that $\mathcal{O}(\Theta)$ is not $A_2+\widetilde{A}_1$, or to prove that Θ has no nozero split Fourier coefficient associated with the unipotent orbit A_2, we apply the same ideas as we did in the case of the orbit B_2. We omit the details.

However, we still have to consider the Fourier coefficients associated with the other representatives of the unipotent orbit A_2. Here we give a local argument. In details, let Θ' denote any irreducible summand of Θ. Let ν be a finite unramified place. As mentioned in the beginning of the proof, the above arguments for the unipotent orbits B_2, $A_2 + \widetilde{A}_1$ and for the split Fourier coefficient corresponding to the unipotent orbit A_2, all work in a similar way for the representation Θ'_ν. In other words we may assume that $\mathcal{O}(\Theta'_\nu)$ is the unipotent orbit A_2 for any unramified place ν. Given a Fourier coefficient of Θ associated to the unipotent orbit A_2, we may choose a place ν such that the stabilizer of the corresponding Jacquet module will be the group SL_3. Arguing as in the global case, using corollary 1, we know that this Jacquet module is zero. Hence, we can deduce that the corresponding Fourier coefficient is zero for all choice of data, and for all representative associated with the unipotent orbit A_2. Thus we may assume that $\mathcal{O}(\Theta)$ is at most $A_1 + \widetilde{A}_1$.

Assume that $\mathcal{O}(\Theta) = A_1 + \widetilde{A}_1$. The set of Fourier coefficients associated with this orbit is described in subsection 2.2. We shall view these Fourier coefficients in an extended way. More precisely, in the notations of subsection 2.2, consider the set of roots $U'_\Delta(1)$. This set consists of 12 roots which are

$$U'_\Delta(1) = \{(0100); (1100); (0110); (1110); (0111); (0120); (0121); (1111);$$
$$(1120); (0122); (1121); (1122)\}$$

The center of the group U_Δ is given by the group $Y = \{x_{1342}(m_1)x_{2342}(m_2)\}$. As can be checked, the quotient U_Δ/Y has a structure of a generalized Heisenberg group. Let $\mathcal{H}_{13}$ denote the Heisenberg group with 13 variables. We view this group as all 13 tuples $(r_1, \ldots, r_6, t_1, \ldots, t_6, z)$ where the product is given as in [I1]. Recall from subsection 2.2 that

the set of Fourier coefficients associated to the unipotent orbit $A_1+\widetilde{A}_1$ are parameterized by a subset of triples $\beta_1,\beta_2,\beta_3\in F^*$. For fixed β_i, the Fourier coefficient is given by integral (7). Define a homomorphism l from U_Δ/Y onto $\mathcal{H}_{13}$ as follows.

$$l(x_{0100}(r_1)x_{0110}(r_2)x_{0111}(r_3)x_{0120}(r_4)x_{0121}(r_5)x_{0122}(r_6))=(r_1,\ldots,r_6,0,\ldots,0)$$

$$l(x_{1100}(t_1)x_{1110}(t_2)x_{1111}(t_3)x_{1120}(t_4)x_{1121}(t_5)x_{1122}(t_6))=(0,\ldots,0,t_1,\ldots,t_6,0)$$

$$l(x_{1220}(z_1)x_{1221}(z_2)x_{1222}(z_3)x_{1231}(z_4)x_{1232}(z_5)x_{1242}(z_6))=(0,\ldots,0,\beta_1z_1+\beta_2z_3+\beta_3z_6)$$

We extend l trivially from U_Δ/Y to U_Δ by $l(Y)=0$. Consider the integral

$$\int\limits_{U_\Delta(F)\backslash U_\Delta(\mathbf{A})}\widetilde{\theta}_\phi^\psi(l(u)g)\theta(ug)du \tag{32}$$

Here $\widetilde{\theta}_\phi^\psi$ is a vector in the theta representation of the group $\mathcal{H}_{13}(\mathbf{A})\cdot\widetilde{Sp}_{12}(\mathbf{A})$. The function ϕ is a Schwartz function of $\mathbf{A}^6$. Arguing as in Lemma 1.1 in [G-R-S3], we deduce that integral (7) is zero for all choice of data if and only if integral (32) is zero for all choice of data. Consider the SL_2 generated by $\{x_{\pm1000}(r)\}$. One can check that this group is inside the stabilizer of the character as defined in integral (7). Hence, if we take $g\in SL_2$, then integral (32) defines an automorphic function in the of this group. It is not hard to check that this copy of SL_2 splits under the double cover when embedded inside $\widetilde{Sp}_{12}$. Indeed, after a suitable conjugation we can embed it inside Sp_{12} as $g\to\mathrm{diag}(g,g,g,g^*,g^*,g^*)$. However, this copy of SL_2 does not split under the double cover of F_4. Therefore, as a function of g, integral (32) defines a genuine automorphic function of $\widetilde{SL}_2(\mathbf{A})$. Our goal is to prove that integral (32) is zero for all choice of data. Since the identity function is not genuine, it follows that integral (32) is zero for all choice of data if and only if, for all $a\in F^*$ the integral

$$\int\limits_{F\backslash\mathbf{A}}\int\limits_{U_\Delta(F)\backslash U_\Delta(\mathbf{A})}\widetilde{\theta}_\phi^\psi(l(u)x_{1000}(r))\theta(ux_{1000}(r))\psi(ar)drdu \tag{33}$$

is zero for all choice of data. Arguing as in Lemma 1.1 in [G-R-S3], integral (33) is zero for all choice of data if and only if the integral

$$\int\limits_{F\backslash\mathbf{A}}\int\limits_{U_\Delta(2)(F)\backslash U_\Delta(2)(\mathbf{A})}\int\limits_{V(F)\backslash V(\mathbf{A})}\theta(vux_{1000}(r))\psi_{U_\Delta(2),u_0}(u)\psi(ar)dvdrdu \tag{34}$$

is zero for all choice of data. Here $U_\Delta(2)$ and $\psi_{U_\Delta(2),u_0}$ are as defined in integral (7). Also, the group V is the unipotent subgroup of F_4 defined by

$$V=\{x_{1100}(t_1)x_{1110}(t_2)x_{1111}(t_3)x_{1120}(t_4)x_{1121}(t_5)x_{1122}(t_6)\}$$

Let $R = P_{\alpha_2,\alpha_3,\alpha_4}$ denote the maximal parabolic subgroup of F_4 whose levi part is GSp_6. Denote its unipotent radical by $U(R)$. Then integral (34) is equal to

$$\int\limits_{U(R)(F)\backslash U(R)(\mathbf{A})} \theta(u)\psi_{U(R)}(u)du \tag{35}$$

where $\psi_{U(R)}$ is defined as follows. We have

$$\psi_{U(R)}(u) = \psi_{U(R)}(x_{1000}(r_1)x_{1220}(r_2)x_{1222}(r_3)x_{1242}(r_4)u') = \psi(ar_1 + \beta_1 r_2 + \beta_2 r_3 + \beta_3 r_4)$$

It follows from the description given in [I] Section 5 that the above Fourier coefficient is associated with the unipotent orbit A_2. Therefore, from the assumption $\mathcal{O}(\Theta) = A_1 + \widetilde{A}_1$, it follows that the integral (35) is zero for all choice of data. Thus, integral (32) is zero for all choice of data and we derived a contradiction. Hence $\mathcal{O}(\Theta)$ is less than the orbit $A_1 + \widetilde{A}_1$.

Finally we consider the case $\mathcal{O}(\Theta) = \widetilde{A}_1$. The set of Fourier coefficients attached to this orbits can be described as follows. Let U'_Δ denote the unipotent group defined by

$$U'_\Delta = \{(0122); (1122); 1222); (1232); (1242); (1342); (2342)\}$$

As before we confuse between a root α and its corresponding one dimensional unipotent group $x_\alpha(r)$. For $\beta \in (F^*)^2\backslash F^*$ we define a character $\psi_{U'_\Delta,\beta}$ of this group as follows. Given $u \in U'_\Delta$ let $\psi_{U'_\Delta,\beta}(x_{1222}(r_1)x_{1242}(r_2)) = \psi(r_1 + \beta r_2)$. Then, the Fourier coefficients associated with this unipotent orbit, are given by

$$\int\limits_{U'_\Delta(F)\backslash U'_\Delta(\mathbf{A})} \theta(u)\psi_{U'_\Delta,\beta}(u)du \tag{36}$$

The stabilizer inside $Spin_7$ of $\psi_{U'_\Delta,\beta}$ contains a unipotent subgroup, for example the group generated by $\{x_{1000}(r)\}$. As in the case of B_2, it is convenient to separate into two cases. First when β is a square, and the second case is when it is not a square. We will consider the first case, and omit the details in the second one.

When β is a square we can conjugate by a suitable element, and integral (36) is zero for all choice of data if and only if the integral

$$\int\limits_{U'_\Delta(F)\backslash U'_\Delta(\mathbf{A})} \theta(u)\psi_{U'_\Delta}(u)du \tag{37}$$

is zero for all choice of data, where now $\psi_{U'_\Delta}(u) = \psi_{U'_\Delta}(x_{1232}(r)u') = \psi(r)$. Arguing in a similar way as in the proof of Lemma 1.1 in [G-R-S3], see also a similar case right before (23), implies that we may consider the integral

$$\int\limits_{U'(F)\backslash U'(\mathbf{A})} \int\limits_{U'_\Delta(F)\backslash U'_\Delta(\mathbf{A})} \theta(u'u)\psi_{U'_\Delta}(u)du'du$$

In other words, integral (37) is zero for all choice of data, provided the above integral is zero for all choice of data. Here U' is the unipotent group which is defined by $U' = \{(0111); (1111); (1221); (1231)\}$. Let $V = U'U'_\Delta$ and define ψ_V to equal $\psi_{U'_\Delta}$ on U'_Δ extended trivially to V. It follows from [C] that the stabilizer of ψ_V is a group of type A_3. It is not hard to check that it is the group SL_4 which contains the abelian unipotent group $Z = \{x_{0120}(m_1)x_{1120}(m_2)x_{1220}(m_3)\}$. Consider the automorphic function of $\widetilde{SL}_4(\mathbf{A})$ defined by

$$f(g) = \int\limits_{V(F)\backslash V(\mathbf{A})} \theta(vg)\psi_V(u)du \tag{38}$$

Since the above group SL_4 does not split under the double cover of F_4, then $f(g)$ is a genuine function. Expand this function along the group Z. The group $SL_3(F)$ embedded in $SL_4(F)$ in the obvious way, acts on this expansion, and we obtain two orbits under this action. Arguing as in the case when $\mathcal{O}(\Theta) = A_1 + \widetilde{A}_1$ we deduce that to prove that integral (38) is zero for all choice of data, it is enough to prove that the integral

$$\int\limits_{(F\backslash \mathbf{A})^3} \int\limits_{V(F)\backslash V(\mathbf{A})} \theta(vx_{0120}(m_1)x_{1120}(m_2)x_{1220}(m_3))\psi_V(u)\psi(m_1)dm_i du \tag{39}$$

is zero for all choice of data. Indeed, if the above integral is zero for all choice of data, then $f(g)$ is equal to its constant term corresponding to a unipotent radical of a maximal parabolic subgroup. This is true only if $f(g)$ is the identity function which is not the case. Using the left invariant property of θ, we have $\theta(h) = \theta(w[214]h)$. Conjugating $w[214]$ in integral (39) from left to right, and exchanging the root (0010) by (1221), we obtain that integral (39) defines a Fourier coefficient associated with the unipotent orbit $A_1 + \widetilde{A}_1$ which is greater than $\mathcal{O}(\Theta) = \widetilde{A}_1$. Hence it is zero, and hence integral (37) is zero for all choice of data. Once again we derived a contradiction.

It follows that Θ has no nonzero Fourier coefficients which corresponds to any unipotent orbit which is greater or equal to $\widetilde{A}_1$. This completes the proof of the Theorem.

$\square$

2.5. **Properties of the Minimal Representation.** In this subsection we shall derive basic properties of the representation Θ. These properties are all a consequence of the smallness properties of this representation.

From Theorem 1 we deduce two important properties of the representation Θ. Let U denote the Heisenberg unipotent radical of F_4. In other words, let $U = U_{\alpha_2,\alpha_3,\alpha_4}$. Let $Z = \{x_{2342}(r)\}$ denote the one dimensional unipotent group attached to the highest root of F_4. Thus, the group Z is the center of U. Define a character ψ_U of $U(F)\backslash U(\mathbf{A})$ as follows.

For $u \in U$, write $u = x_{1000}(r)u'$. Define $\psi_U(u) = \psi(r)$. (See subsection 2.1) For any $g \in F_4(\mathbf{A})$, denote

$$\theta^{U,\psi}(g) = \int\limits_{U(F)\backslash U(\mathbf{A})} \theta(ug)\psi_U(u)du$$

Similarly, we denote

$$\theta^{U}(g) = \int\limits_{U(F)\backslash U(\mathbf{A})} \theta(ug)du$$

From Theorem 1 we deduce

Proposition 3. *With the above notations, we have the following expansion*

$$\int\limits_{Z(F)\backslash Z(\mathbf{A})} \theta(zg)dz = \theta^U(g) + \sum_{\gamma\in Q(F)\backslash Sp_6(F)} \sum_{\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)\gamma g) \tag{40}$$

Here Q is the maximal parabolic subgroup of Sp_6, whose Levi part is the group GL_3.

Proof. The group $Z\backslash U$ is abelian. Hence, we have the following Fourier expansion

$$\int\limits_{Z(F)\backslash Z(\mathbf{A})} \theta(zg)dz = \sum_{\gamma\in L(F)} \int\limits_{U(F)\backslash U(\mathbf{A})} \theta(ug)\psi_\gamma(u)du$$

where $L(F)$ runs over all characters of $Z(\mathbf{A})U(F)\backslash U(\mathbf{A})$. We can identify the group $L(F)$ with $F^{14} \simeq U(F)/Z(F)$. The group $Sp_6(F)$ acts on $L(F)$ as the third fundamental representation of Sp_6. We have three type of orbits. First, we have the orbit corresponding to the zero vector. Then, we have the orbit generated by the group $\{x_{1000}\}$. The third type of orbits, are all the other ones not included in the first two. It is not hard to show that the Fourier coefficients which corresponds to an orbit of the third type correspond to a unipotent orbit which is greater than the unipotent orbit A_1. By Theorem 1 they contribute zero to the above expansion. Thus we are left with the first two type of orbits. The trivial orbit corresponds to the constant term, and the second one corresponds to the Fourier coefficient $\theta^{U,\psi}$. From this expansion (40) follows. $\square$

Another result which can be derived from Theorem 1 is the following. Let U_Q denote the unipotent radical of Q where Q is the parabolic subgroup of Sp_6 which was defined in Proposition 3. Let Q^0 denote the subgroup of Q defined by $Q^0 = SL_3 \cdot U_Q$. We have

Proposition 4. *For all $q \in Q^0(\mathbf{A})$, we have*

$$\theta^{U,\psi}(qg) = \theta^{U,\psi}(g). \tag{41}$$

Proof. Let U_{Sp_6} denote the maximal unipotent subgroup of Sp_6. The group $Q^0(\mathbf{A})$ is generated by $U_{Sp_6}(\mathbf{A})$ and the two simple reflections $w[3]$ and $w[4]$. Clearly (41) holds for the

above two simple reflections. Thus its enough to prove (41) for $q \in U_{Sp_6}(\mathbf{A})$. The group U_Q is abelian. Hence we can consider the Fourier expansion of $\theta^{U,\psi}$ along this group. We have

$$\theta^{U,\psi}(g) = \sum_\gamma \int\limits_{U_Q(F)\backslash U_Q(\mathbf{A})} \theta^{U,\psi}(vg)\psi_\gamma(v)dv$$

where we sum over all characters of the group $U_Q(F)\backslash U_Q(\mathbf{A})$. We claim that for all nontrivial characters, the Fourier coefficient

$$\int\limits_{U_Q(F)\backslash U_Q(\mathbf{A})} \theta^{U,\psi}(vg)\psi_\gamma(v)dv$$

is zero for all choice of data. This follows from the same type of arguments as in the proof of Theorem 1. Indeed, when considering suitable Fourier expansions of the above integral we obtain two types of integrals. The first type are Fourier coefficients which are associated with unipotent orbits which are greater than A_1. Hence, by Theorem 1 they are zero. The second type is an integral of the form

$$\int\limits_{Y(F)\backslash Y(\mathbf{A})} \theta^{U(R)}(y)\psi_Y(y)dy$$

Here R is a certain maximal parabolic subgroup of F_4 and $U(R)$ is its unipotent radical. The group Y is a subgroup of $M(R)$, the Levi part of R. Finally, the character ψ_Y is associated with a unipotent orbit which is greater than the minimal orbit of $M(R)$. Thus, from Proposition 1 this integral is zero for all choice of data.

Hence, only the constant term remains, and we proved (41) for all $q \in U_Q(\mathbf{A})$. In a similar way, using again Proposition 1, we obtain the invariance property of $\theta^{U,\psi}$ along the adelic points of U_{Sp_6}/U_Q. $\square$

The next Proposition relate the minimal representation of $\widetilde{F}_4$ to the theta representation defined on the symplectic group $\widetilde{Sp}_{14}$. Consider the Fourier coefficient corresponding to the unipotent orbit A_1. In other words, consider the integral

$$\theta^{Z,\psi_\beta}(g) = \int\limits_{F\backslash \mathbf{A}} \theta(x_{2342}(r)g)\psi(\beta r)dr$$

Here $\beta \in F^*$. This Fourier coefficient defines an automorphic representation of $\widetilde{Sp}_6(\mathbf{A})$. Let $\mathcal{H}_{15}$ denote the Heisenberg group with 15 variables. The group U is isomorphic to $\mathcal{H}_{15}$. We shall denote this isomorphism by ι. We have

Proposition 5. *With the above notations, the space of functions*

$$\theta_{Sp_{14}}^{\phi,\psi_\beta}(\iota(u)\varpi_3(g))$$

is a dense subspace in the space of functions $\theta^{Z,\psi_\beta}(ug)$. Here $g \in \widetilde{Sp}_6(\mathbf{A})$, $u \in U(\mathbf{A})$ and $\theta_{Sp_{14}}^{\phi,\psi_\beta} \in \Theta_{Sp_{14}}^{\phi,\psi_\beta}$ is the theta representation of $\mathcal{H}_{15}(\mathbf{A}) \cdot \widetilde{Sp}_{14}(\mathbf{A})$ attached to the character ψ_β. Also, we denote by ϖ_3 the third fundamental representation of Sp_6.

Proof. It follows from [I1] that the space of functions

$$\theta_{Sp_{14}}^{\phi,\psi_\beta}(\iota(u)\varpi_3(g)) \int\limits_{U(F)\backslash U(\mathbf{A})} \theta_{Sp_{14}}^{\phi',\psi_\beta}(\iota(v)\varpi_3(g))\theta(vg)dv$$

is a dense subspace in the space of functions $\theta^{Z,\psi_\beta}(ug)$. The result will follows once we prove that as a function of $g \in Sp_6(\mathbf{A})$, the integral

$$\int\limits_{U(F)\backslash U(\mathbf{A})} \theta_{Sp_{14}}^{\phi',\psi_\beta}(\iota(v)\varpi_3(g))\theta(vg)dv$$

is the identity function. Since the embedding of Sp_6 in both Sp_{14} via the third fundamental representation does not split under the double cover, we deduce that the above integral is not a genuine function. Hence, to obtain the result, it is enough to prove that for all $a \in F^*$ the integral

$$\int\limits_{U(F)\backslash U(\mathbf{A})} \int\limits_{F\backslash \mathbf{A}} \theta_{Sp_{14}}^{\phi',\psi_\beta}(\iota(v)\varpi_3(x_{0122}(r)))\theta(vx_{0122}(r))\psi(ar)drdv$$

is zero for all choice of data. Unfolding the theta function, we obtain as an inner integration the integral

$$\int\limits_{V(F)\backslash V(\mathbf{A})} \theta(v)\psi_V(v)dv$$

Here, the group V is the unipotent subgroup of F_4 which is associated with the seven positive roots of F_4 of the form $(n_1n_2n_3n_4)$ with $n_4 = 2$. The character ψ_V is defined as $\psi_V(v) = \psi_V(x_{0122}(r_1)x_{2342}(r_2)v') = \psi(ar_1 + \beta r_2)$. Thus, the above integral is a Fourier coefficient which is associated with the unipotent orbit $\widetilde{A}_1$. From Theorem 1 it is zero for all choice of data.

□

2.6. **On Minimal Representations of the Group $\widetilde{Sp}_6(\mathbf{A})$.** Let $\Theta_{Sp_6}^{(2)}$ denote a minimal representation of $\widetilde{Sp}_6(\mathbf{A})$. By definition this means that given any unipotent orbit of Sp_6 which is greater than (21^4), then all Fourier coefficients of $\Theta_{Sp_6}^{(2)}$ which are associated with this orbit (see [G1]) are zero for all choice of data. In the computations we shall perform we will need for the representation $\Theta_{Sp_6}^{(2)}$, similar properties to the ones we stated and proved in subsection 2.5. More precisely, we will need analogous results to those which are stated in Propositions 1, 3, 4 and 5.

Recall that Sp_6 has three maximal parabolic subgroups. Let $P(GL_3)$ denote the maximal parabolic subgroup of Sp_6 whose Levi part is GL_3. Similarly, we shall denote the other two maximal parabolic subgroups by $P(GL_2 \times SL_2)$ and $P(GL_1 \times Sp_4)$. We denote by $U(GL_3)$ the unipotent radical of $P(GL_3)$, and use similar notations for the other two maximal parabolic subgroups. We remark that the group GL_3 embedded in Sp_6 as the Levi part of $P(GL_3)$, splits under the double cover of Sp_6. To prove the analogous Proposition to Proposition 1, we define the group $\widetilde{M}_0$ for each maximal parabolic subgroup P. When $P = P(GL_3)$ we denote $\widetilde{M}_0 = GL_3$. When $P = P(GL_2 \times SL_2)$ we define $\widetilde{M}_0 = GL_2 \times \widetilde{SL}_2$, and when $P = P(GL_1 \times Sp_4)$ we denote $\widetilde{M}_0 = \widetilde{Sp}_4$. When $\widetilde{M}_0 = GL_3$, a representation of $\widetilde{M}_0(\mathbf{A})$ is said to be minimal if it is one dimensional. When $\widetilde{M}_0 = GL_2 \times \widetilde{SL}_2$, a representation of $\widetilde{M}_0(\mathbf{A})$ is said to be minimal if it is one dimensional on GL_2. Finally, when $\widetilde{M}_0 = \widetilde{Sp}_4$, a representation of $\widetilde{M}_0(\mathbf{A})$ is said to be minimal if it is a minimal representation of $\widetilde{Sp}_4$, that is its only nonzero Fourier coefficients are associated with the unipotent orbit (21^2) of Sp_4. We start with

Proposition 6. *Let U denote any unipotent radical of a maximal parabolic subgroup of Sp_6. Then, as a representation of $\widetilde{M}_0(\mathbf{A})$, the constant term $\Theta_{Sp_6}^{(2),U}$ is a minimal representation.*

Proof. Consider the case when U us the unipotent radical of $P(GL_3)$. In this case, consider the one dimensional unipotent subgroup $N = \{x(r) = I_6 + r(e_{1,3} - e_{4,6})\}$. Here $e_{i,j}$ is the matrix of size six which has a one at the (i,j) entry and zero otherwise. Expand the constant term $\Theta_{Sp_6}^{(2),U}$ along the group $N(F)\backslash N(\mathbf{A})$. We claim that for all $a \in F^*$, the integral

$$\int\limits_{F\backslash \mathbf{A}} \theta_{Sp_6}^{(2),U}(x(r))\psi(ar)dr$$

is zero for all choice of data. Here $\theta_{Sp_6}^{(2)}$ is a vector in the space of $\Theta_{Sp_6}^{(2)}$. Indeed, in this case the above integral contains as an inner integration a Fourier coefficient which corresponds to the unipotent orbit $(2^2 1^2)$. Since $\Theta_{Sp_6}^{(2)}$ is a minimal representation, these Fourier coefficients are all zero. This means that as a function of $GL_3(\mathbf{A})$, the constant term $\Theta_{Sp_6}^{(2),U}$ is invariant under a copy of $SL_2(\mathbf{A})$. Thus, as a function of $GL_3(\mathbf{A})$, the constant term $\Theta_{Sp_6}^{(2),U}$ is a one dimensional representation.

The other two maximal parabolic subgroups are treated in the same way.

□

The next Proposition is the Sp_6 version of Propositions 3 and 4. Let U denote the unipotent radical of the parabolic subgroup $P(GL_3)$. In terms of matrices we can identify U with all matrices of the form $\begin{pmatrix} I & X \\ & I \end{pmatrix}$ where $I = I_3$ and $X \in Mat_3^0 = \{X \in Mat_3 : X =$

$J_3X^tJ_3\}$. Let ψ_U be defied as

$$\psi_U(u) = \psi_U\left(\begin{pmatrix} I_3 & X \\ & I_3 \end{pmatrix}\right) = \psi(x_{3,1})$$

and denote

$$\theta_{Sp_6}^{(2),U,\psi}(g) = \int\limits_{U(F)\backslash U(\mathbf{A})} \theta_{Sp_6}^{(2)}(ug)\psi_U(u)du$$

If we embed the group GL_3 inside Sp_6 as $g \mapsto \text{diag}(g, g^*)$, then the stabilizer of ψ_U inside GL_3 is the group of all matrices of the form

$$L_0(GL_3) = \left\{ \begin{pmatrix} h & y \\ & 1 \end{pmatrix} \quad h \in GL_2, \quad y \in Mat_{2\times 1} \right\}$$

Let $L(GL_3)$ denote the maximal parabolic subgroup of GL_3 which contains $L_0(GL_3)$. Finally, let $L^0(GL_3)$ denote the subgroup of $L_0(GL_3)$ such that $h \in SL_2$. With these notations we prove

Proposition 7. *We have the following expansion,*

$$\theta_{Sp_6}^{(2)}(g) = \theta_{Sp_6}^{(2),U}(g) + \sum_{\gamma\in L(GL_3)(F)\backslash GL_3(F)} \sum_{\epsilon\in\{\pm 1\}\backslash F^*} \theta_{Sp_6}^{(2),U,\psi}(h(\epsilon)\gamma g) \tag{42}$$

Here $h(\epsilon) = diag(I_2, \epsilon, \epsilon^{-1}, I_2)$. Moreover we have

$$\theta_{Sp_6}^{(2),U,\psi}(qg) = \theta_{Sp_6}^{(2),U,\psi}(g) \tag{43}$$

for all $q \in L^0(GL_3)(\mathbf{A})$.

Proof. The proof is similar to the proof of Propositions 3 and 4. Notice that U is an abelian group. Therefore, we can expand $\theta_{Sp_6}^{(2)}(g)$ along $U(F)\backslash U(\mathbf{A})$. The group $GL_3(F)$ acts on the character group of $U(F)\backslash U(\mathbf{A})$, and all characters except the trivial one and any character that is in the same orbit of ψ_U, contribute zero to the expansion. This follows from the fact that any other character produces a Fourier coefficient which is associated with a unipotent orbit which is greater than (21^4). From this, identity (42) follows.

As for identity (43), it follows from similar arguments. Indeed, let $N = \{I_6 + r_1(e_{1,2} - e_{5,6}) + r_2(e_{1,3} - e_{4,6})\}$. Expanding $\theta_{Sp_6}^{(2),U,\psi}(g)$ along $N(F)\backslash N(\mathbf{A})$, it follows from the fact that $\Theta_{Sp_6}^{(2)}$ is a minimal representation, that nontrivial characters of $N(F)\backslash N(\mathbf{A})$ contributes zero to the expansion. Thus $\theta_{Sp_6}^{(2),U,\psi}(g) = \theta_{Sp_6}^{(2),UN,\psi}(g)$. Since $L^0(GL_3)(\mathbf{A})$ is generated by $N(\mathbf{A})$ and the Weyl element $\text{diag}(J_2, I_2, J_2)$, identity (43) follows.

□

Finally, we prove the analogous to Proposition 5. To do that, let Z denote the unipotent subgroup defined by $Z = \{x(r) = I_6 + re_{1,6}\}$. For $\beta \in F^*$, denote

$$\theta_{Sp_6}^{(2),Z,\psi_\beta}(g) = \int\limits_{F\backslash \mathbf{A}} \theta_{Sp_6}^{(2)}(x(r)g)\psi(\beta r)dr$$

Let U denote the unipotent radical of the maximal parabolic subgroup $P(GL_1 \times Sp_4)$. Then U can be identified with the Heisenberg group $\mathcal{H}_5$. As in Proposition 5 we have

Proposition 8. *With the above notations, the space of functions*

$$\theta_{Sp_4}^{\phi,\psi_\beta}(ug)$$

is a dense subspace in the space of functions $\theta_{Sp_6}^{(2),Z,\psi_\beta}(ug)$. *Here* $g \in \widetilde{Sp}_4(\mathbf{A})$, $u \in U(\mathbf{A})$ *and* $\theta_{Sp_4}^{\phi,\psi_\beta} \in \Theta_{Sp_4}^{\phi,\psi_\beta}$ *is the theta representation of* $\mathcal{H}_5(\mathbf{A}) \cdot \widetilde{Sp}_4(\mathbf{A})$ *attached to the character* ψ_β.

3. Commuting Pairs in F_4

Let (H, G) be a commuting pair in the group F_4. By that we mean that the two groups commute one with the other, but they need not be a dual pair. Let $\mathcal{E}$ denote an automorphic representation of the group $F_4(\mathbf{A})$. Let π denote an irreducible cuspidal representation of $H(\mathbf{A})$, and let

$$f(g) = \int\limits_{H(F)\backslash H(\mathbf{A})} \varphi_\pi(h)E((h,g))dh \tag{44}$$

Here E is a vector in the space of $\mathcal{E}$, and φ_π is a vector in the space of π. Denote by $\sigma(\pi, \mathcal{E})$ the automorphic representation of $G(\mathbf{A})$ generated by all the functions $f(g)$ defined above.

As explained in the introduction we are looking for those cases which satisfy equation (4). In this case, since V is trivial, equation (4) is given by

$$\dim \pi + \dim \mathcal{E} = \dim H + \dim \sigma(\pi, \mathcal{E}) \tag{45}$$

We will consider the following commuting pairs:

1) $(H, G) = (SL_3, SL_3)$.

2) $(H, G) = (SL_2 \times SL_2, Sp_4)$.

3) $(H, G) = (SL_2, SL_4)$.

4) $(H, G) = (SO_3, G_2)$.

5) $(H, G) = (SL_2, Sp_6)$.

The way these groups are embedded inside F_4 will be discussed below. In each of the above cases we check the conditions such that equation (45) holds. Notice that in integral (44), there is a symmetry between H and G. In other words, given an irreducible cuspidal

representation σ of the group $G(\mathbf{A})$, we can consider the representation of $H(\mathbf{A})$ generated by the space of functions

$$\int\limits_{G(F)\backslash G(\mathbf{A})} \varphi_\sigma(h)E((h,g))dg \tag{46}$$

The corresponding equation for this case is

$$\dim\,\sigma + \dim\,\mathcal{E} = \dim\,G + \dim\,\pi(\sigma,\mathcal{E}) \tag{47}$$

Thus, in each of the above cases we should check both options. The representation $\mathcal{E}$ is defined on F_4, and hence its dimension should be a half of the dimension of some unipotent orbit of F_4. For a list of the unipotent orbits, and their dimensions, we refer the reader to [C-M] page 128. It follows from that list that the minimal representation, the one constructed in the previous Section, is of dimension 8. The one above it is of dimension 11, and so on. We have

1) $(H,G) = (SL_3, SL_3)$. Since π is cuspidal, then it is generic, and hence $\dim\,\pi = 3$. We have $\dim\,SL_3 = 8$. Hence, equation (45) is $\dim\,\mathcal{E} - \dim\,\sigma(\pi,\mathcal{E}) = 5$. Since $\sigma(\pi,\mathcal{E})$ is an automorphic representation of SL_3, its dimension is at most 3, and hence the only option is that $\dim\,\mathcal{E} = 8$ and $\dim\,\sigma(\pi,\mathcal{E}) = 3$.

2) $(H,G) = (SL_2 \times SL_2, Sp_4)$. Here $\dim\,H = 6$, and $\dim\,\pi = 2$. Hence we have $\dim\,\mathcal{E} - \dim\,\sigma(\pi,\mathcal{E}) = 4$. The representation $\sigma(\pi,\mathcal{E})$ is an automorphic representation of Sp_4, hence its dimension is 2,3 or 4. Thus, the only option is $\dim\,\mathcal{E} = 8$ and $\dim\,\sigma(\pi,\mathcal{E}) = 4$. Thus we expect $\sigma(\pi,\mathcal{E})$ to be generic.

To consider the options for integral (46) we notice that $\dim\,G = \dim\,Sp_4 = 10$, and since $\pi(\sigma,\mathcal{E})$ is an automorphic representation on $SL_2(\mathbf{A}) \times SL_2(\mathbf{A})$, then $\dim\,\pi(\sigma,\mathcal{E}) = 1, 2$. Thus, we have two options, first $12 = \dim\,\mathcal{E} + \dim\,\sigma$ and the second is $11 = \dim\,\mathcal{E} + \dim\,\sigma$. The representation σ is a cuspidal representation on Sp_4, and hence its dimension is at most 4. Thus in both cases we have $\dim\,\mathcal{E} = 8$. In the first case we get $\dim\,\sigma = 4$ and in the second $\dim\,\sigma = 3$.

3) $(H,G) = (SL_2, SL_4)$. Since $\dim\,H = 3$ and $\dim\,\pi = 1$, we obtain $\dim\,\mathcal{E} - \dim\,\sigma(\pi,\mathcal{E}) = 2$. Thus, the only option is $\dim\,\mathcal{E} = 8$ and $\dim\,\sigma(\pi,\mathcal{E}) = 6$.

In the other direction, we have $\dim\,G = \dim\,SL_4 = 15$. Also, since σ is cuspidal, it must be generic, and hence $\dim\,\sigma = 6$. The group $H = SL_2$, and hence $\dim\,\pi(\sigma,\mathcal{E}) = 1$. Thus we obtain $15 + 1 = \dim\,\mathcal{E} + 6$, or $\dim\,\mathcal{E} = 10$. From [C-M] it follows that there is no unipotent orbit whose dimension is 20, and hence we dont expect a representation of F_4 whose dimension is 10.

4) $(H, G) = (SO_3, G_2)$. As in the previous case we obtain dim $\mathcal{E} -$ dim $\sigma(\pi, \mathcal{E}) = 2$. Thus, the only option is dim $\mathcal{E} = 8$ and dim $\sigma(\pi, \mathcal{E}) = 6$. Hence, we expect the image of this lift to be a generic representation of G_2.

In the other direction we have dim $G =$ dim $G_2 = 14$. Since $H = SO_3$, then dim $\pi(\sigma, \mathcal{E}) = 1$. Also, σ is a cuspidal representation of G_2, and hence dim $\sigma = 5, 6$. This implies that $14 + 1 =$ dim $\mathcal{E} +$ dim σ, and hence dim $\mathcal{E} = 9, 10$. By [C-M] we dont expect a representation which such dimensions.

5) $(H, G) = (SL_2, Sp_6)$. Here H is of the same type as the previous two cases, and hence we get the identity dim $\mathcal{E} -$ dim $\sigma(\pi, \mathcal{E}) = 2$. Since $\sigma(\pi, \mathcal{E})$ is a representation of Sp_6, its dimension is at most 9. Thus dim $\mathcal{E}$ is at most 11, and there are two cases. First, when dim $\mathcal{E} = 11$ and then dim $\sigma(\pi, \mathcal{E}) = 9$. In this case $\sigma(\pi, \mathcal{E})$ is a generic representation. The second case is when dim $\mathcal{E} = 8$ and dim $\sigma(\pi, \mathcal{E}) = 6$.

In the other direction, since dim $G =$ dim $Sp_6 = 21$, and $H = SL_2$ then dim $\pi(\sigma, \mathcal{E}) = 1$, and hence dim $\mathcal{E} +$ dim $\sigma = 22$. The representation σ is cuspidal, and hence dim $\sigma = 6, 8, 9$. From this we obtain that dim $\mathcal{E} = 15, 14, 13$. From [C-M] we deduce that the last case is impossible, but it is possible that dim $\mathcal{E} = 15, 14$.

As can be seen from the above in all cases, except case number **5)**, the only representation $\mathcal{E}$ of F_4 which satisfies the dimension equations (45) or (47) is the minimal representation Θ. In the following subsections we shall consider the above cases. In each case we will determine when the image of the lift is cuspidal and when it is nonzero. We will consider both liftings given by integrals (44) and (46) even though the dimension formula may not work in both directions. We do that since studying the other direction as well may give us some information of how to characterize the image of the lift. In this paper we only consider the case when $\mathcal{E} = \Theta$, the minimal representation of the double cover of F_4. This implies that some of the representations are defined on the double cover of H or G.

3.1. **The Commuting Pair** (SL_3, SL_3)**.** In this subsection we will study the lifting from the double cover of GL_3 to the linear group SL_3, and the lift from GL_3 to the double cover of SL_3. We shall denote by $\widetilde{SL}_3$ the double cover of SL_3, and similarly for GL_3.

3.1.1. **From** $\widetilde{GL}_3$ **to** SL_3**.** To construct this lifting, we first embed the commuting pair (SL_3, SL_3) inside F_4 as follows. The first copy of SL_3 is generated by $< x_{\pm(1000)}(r_1), x_{\pm(0100)}(r_2), x_{\pm(1100)}(r_3) >$ and the other copy is generated by $< x_{\pm(0001)}(r_1), x_{\pm(1231)}(r_2), x_{\pm(1232)}(r_3) >$. Notice that the first copy is generated by unipotent elements which corresponds to long roots, and the second copy by unipotent elements corresponding to short roots. This means that the first copy of SL_3, when embedded as above inside F_4, does not split under the covering, but the second copy does.

Let $\widetilde{\pi}$ denote a cuspidal representation of the group $\widetilde{GL}_3(\mathbf{A})$. We consider the integral

$$f(h) = \int\limits_{SL_3(F)\backslash SL_3(\mathbf{A})} \widetilde{\varphi}(g)\theta((h,g))dg \tag{48}$$

Here $\widetilde{\varphi}$ is a vector in the space of $\widetilde{\pi}$ and $(h,g) \in (SL_3(\mathbf{A}), SL_3(\mathbf{A}))$ embedded in $\widetilde{F}_4(\mathbf{A})$ as above. In other words, the first copy of SL_3 is the one which is generated by short roots in F_4, and the second copy is equal to $< x_{\pm 1000}(r), x_{\pm 0100}(r) >$. The function $f(h)$ defines an automorphic function of $SL_3(\mathbf{A})$. As we vary the data in integral (48), we obtain an automorphic representation of $SL_3(\mathbf{A})$ which we shall denote by $\sigma(\widetilde{\pi})$. Our first result is

Proposition 9. *The representation $\sigma(\widetilde{\pi})$ is a nonzero cuspidal representation of $SL_3(\mathbf{A})$.*

Proof. To prove cuspidality, we have to show that the integrals

$$I = \int\limits_{V(F)\backslash V(\mathbf{A})} f(vh)dv$$

is zero for all choice of data, where V is any unipotent radical of a maximal parabolic subgroup of SL_3. Up to conjugation there are two such unipotent radicals. They are given by $V_1 = \{x_{0001}(r_1)x_{1232}(r_2)\}$ and $V_2 = \{x_{1231}(r_1)x_{1232}(r_2)\}$. It is easy to see that the Weyl element $w[321323]$ conjugates V_1 to V_2 and fixes the group $SL_3 =< x_{\pm(1000)}(r_1), x_{\pm(0100)}(r_2) >$. Hence, to prove the cuspidality of $\sigma(\widetilde{\pi})$, it is enough to show that the constant term of $f(h)$ along $V = V_2$, is zero for all choice of data.

Let U_1 denote the unipotent subgroup of F_4 generated by all $< x_\alpha(r) >$ where $\alpha \in \{0122; 1122; 1222; 1242; 1342; 2342\}$. Let $U_2 =< U_1, x_{1232}(r) >$. We expand I along the group $U_1(F)\backslash U_1(\mathbf{A})$. The group $Spin_6(F)$ generated by $< x_{\pm(1000)}(r); x_{\pm(0100)}(r); x_{\pm(0120)}(r) >$ acts on this expansion with three type of orbits. The first type of orbit correspond to the set of all vectors in F^6 which have nonzero length. Combining the integration over $U_1(F)\backslash U_1(\mathbf{A})$ with the integration over $x_{1232}(r)$ we obtain the integral

$$\int\limits_{U_2(F)\backslash U_2(\mathbf{A})} \theta(u_2 m)\psi(\gamma \cdot u_2)du_2$$

as an inner integration to the expansion. Here $\gamma \in F^7$ is a vector with a nonzero length. However, this Fourier coefficient corresponds to the unipotent orbit $\widetilde{A}_1$. By the minimality of Θ it is zero. Hence we are left with the two orbits which corresponds to the zero vector and to all nonzero vectors with zero length. Thus I is equal to

$$\int\limits_{SL_3(F)\backslash SL_3(\mathbf{A})} \int\limits_{F\backslash \mathbf{A}} \int\limits_{U_2(F)\backslash U_2(\mathbf{A})} \widetilde{\varphi}(g)\theta(u_2(x_{1231}(r_1), g))du_2 dr_1 dg +$$

$$\int\limits_{SL_3(F)\backslash SL_3(\mathbf{A})} \widetilde{\varphi}(g) \int\limits_{F\backslash \mathbf{A}} \int\limits_{U_2(F)\backslash U_2(\mathbf{A})} \sum_{\gamma\in S(F)\backslash Spin_6(F)} \theta(u_2\gamma(x_{1231}(r_1),g))\psi_{U_2}(u_2)du_2dr_1dg$$

where ψ_{U_2} is defined as follows. If $u_2 = x_{0122}(r_1)u_2'$, then define $\psi_{U_2}(u_2) = \psi(r_1)$. (See subsection 2.1 for notations). Also, the group S is the stabilizer of ψ_{U_2} inside $Spin_6$. Thus

$$S =< x_{\pm(0100)}(r); x_{\pm(0120)}(r); x_{1000}(r); x_{1100}(r); x_{1120}(r); x_{1220}(r) >$$

Denote the first summand by I' and the second one by I''.

We start with I''. Let L denote the maximal parabolic subgroup of $Spin_6$ which contains the copy of SL_3 generated by $< x_{\pm(1000)}(r_1), x_{\pm(0100)}(r_2) >$. The space $S(F)\backslash Spin_6(F)/L(F)$ contains two representatives which can be chosen to be e and $w[1323]$. Thus, I'' is equal to

$$\int\limits_{S(2)(F)\backslash SL_3(\mathbf{A})} \int\limits_{F\backslash \mathbf{A}} \int\limits_{U_2(F)\backslash U_2(\mathbf{A})} \widetilde{\varphi}(g)\theta(u_2(x_{1231}(r_1),g))\psi_{U_2}(u_2)du_2dr_1dg+$$

$$\int\limits_{S(1)(F)\backslash SL_3(\mathbf{A})} \widetilde{\varphi}(g) \int \sum_{\delta_i\in F} \theta(u_2w[123]x_{0120}(\delta_1)x_{1120}(\delta_2)(x_{1231}(r_1),g))\psi_{U_2}(u_2)du_2dr_1dg$$

where we used the left invariant of θ under rational points to replace the Weyl element $w[1323]$ by $w[123]$. Here, the group $S(1)$ denotes the maximal parabolic subgroup of SL_3 which contains the group $\{x_{\pm 1000}\}$. Similarly we define $S(2)$. Also, in the second summand, the variables r_1 and u_2 are integrated as in the first summand. Denote the first summand by I_1'' and the second one by I_2''. We start with I_1''. Expand it along the group U/Z with points in $F\backslash\mathbf{A}$. Here $U = U_{\alpha_2,\alpha_3,\alpha_4}$ is the unipotent radical of the maximal parabolic subgroup of F_4 whose Levi part is GSp_6, and $Z = \{x_{2342}(m)\}$ is its center. Using Proposition 3, this expansion contains two summands. The constant term in the expansion of I_1'' contributes zero to the integral. Indeed, it is equal to

$$\int\limits_{S(2)(F)\backslash SL_3(\mathbf{A})} \int\limits_{F\backslash \mathbf{A}} \int\limits_{Z(\mathbf{A})U_2(F)\backslash U_2(\mathbf{A})} \widetilde{\varphi}(g)\theta^U(u_2(x_{1231}(r_1),g))\psi_{U_2}(u_2)du_2dr_1dg$$

The unipotent radical of $S(2)$ is the unipotent group $L = \{x_{1000}(m_1)x_{1100}(m_2)\}$. Notice that L is a subgroup of U. Hence, as a function of g, the integral

$$\int\limits_{F\backslash \mathbf{A}} \int\limits_{Z(\mathbf{A})U_2(F)\backslash U_2(\mathbf{A})} \theta^U(u_2(x_{1231}(r_1),g))\psi_{U_2}(u_2)du_2dr_1$$

is left invariant under $l \in L(\mathbf{A})$. Hence, we get the integral $\int\limits_{L(F)\backslash L(\mathbf{A})} \widetilde{\varphi}(lg)dl$ as inner integration. This integral is zero by the cuspidality of $\widetilde{\pi}$.

Thus I_1'' is equal to

$$\int\limits_{S(2)(F)\backslash SL_3(\mathbf{A})} \widetilde{\varphi}(g) \int \sum_{\gamma\in Q(F)\backslash Sp_6(F)} \sum_{\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)\gamma u_2(x_{1231}(r_1),g))\psi_{U_2}(u_2)du_2dr_1dg$$

where r_1 is integrated as before and u_2 is integrated over $Z(\mathbf{A})U_2(F)\backslash U_2(\mathbf{A})$. Let P denote the maximal parabolic subgroup of Sp_6 whose Levi part contains Sp_4. The space $Q(F)\backslash Sp_6(F)/P(F)$ consists of two elements and as representatives we choose e and $w[234]$. Hence, I_1'' is equal to

$$\int\limits_{S(2)(F)\backslash SL_3(\mathbf{A})} \widetilde{\varphi}(g) \int \sum_{\gamma\in S(3)(F)\backslash Sp_4(F)} \sum_{\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)\gamma u_2(x_{1231}(r_1),g))\psi_{U_2}(u_2)du_2dr_1dg+$$

$$\int \widetilde{\varphi}(g) \sum_{\gamma\in S(3)(F)\backslash Sp_4(F)} \sum_{\delta_i\in F;\ \epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[234]y(\delta_1,\delta_2,\delta_3)\gamma u_2(x_{1231}(r_1),g))\psi_{U_2}(u_2)du_2dr_1dg$$

where all variables in the second summand are integrated as in the first summand. Also, we have $y(\delta_1,\delta_2,\delta_3)=x_{0001}(\delta_1)x_{0011}(\delta_2)x_{0122}(\delta_3)$. Notice that $x_{0122}(r)$ commutes with $\gamma\in Sp_4$ and that this group actually normalizes the group U_2 . Hence, in the first summand, we can conjugate this unipotent element to the left, and using Proposition 4, we deduce that

$$g\mapsto \theta^{U,\psi}(h_2(\epsilon)\gamma x_{0122}(r)u_2(x_{1231}(r_1),g))$$

is left invariant by $x_{0122}(r)$ for all $r\in\mathbf{A}$. Since ψ_{U_2} is nontrivial on $x_{0122}(r)$, the first summand is zero. In the second summand, after conjugating u_2 across γ, we conjugate the unipotent element $x_{1122}(r)$ to the left. We have

$$h_2(\epsilon)w[234]y(\delta_1,\delta_2,\delta_3)x_{1122}(r)=x_{1000}(\epsilon^{-1}r)h_2(\epsilon)w[234]y(\delta_1,\delta_2,\delta_3)$$

Changing variables, we obtain $\int\limits_{F\backslash\mathbf{A}}\psi(\epsilon^{-1}r)dr$ as inner integration. This integral is clearly zero, and hence $I_1''=0$.

Next we consider I_2''. Expanding along U/Z, using Proposition 3, the nontrivial orbit contributes

$$\int\limits_{S(1)(F)\backslash SL_3(\mathbf{A})} \int\limits_{F\backslash\mathbf{A}} \int\limits_{Z(\mathbf{A})U_2(F)\backslash U_2(\mathbf{A})} \widetilde{\varphi}(g)\times$$

$$\sum_{\gamma\in Q(F)\backslash Sp_6(F)} \sum_{\delta_i\in F,\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)\gamma u_2w[123]x_{0120}(\delta_1)x_{1120}(\delta_2)(x_{1231}(r_1),g))\psi_{U_2}(u_2)du_2dr_1dg$$

We consider the space $Q(F)\backslash Sp_6(F)/P(F)$. Arguing as in the computation of I_1'' we obtain that this integral is zero. Thus we are left with the contribution from the constant term

$$\int\limits_{S(1)(F)\backslash SL_3(\mathbf{A})} \int\limits_{(F\backslash\mathbf{A})^2} \widetilde{\varphi}(g)\sum_{\delta_i\in F}\theta^{U}(x_{0122}(r)w[123]x_{0120}(\delta_1)x_{1120}(\delta_2)(x_{1231}(r_1),g))\psi(r)drdr_1dg$$

Conjugate $x_{1231}(r_1)$ to the left. We obtain the integral

$$\int\limits_{(F\backslash \mathbf{A})^2} \sum_{\delta_i\in F} \theta^U(x_{0121}(r_1)x_{0122}(r)w[123]x_{0120}(\delta_1)x_{1120}(\delta_2)(1,g))\psi(r)drdr_1 \tag{49}$$

as inner integration. Expand this integral along the unipotent element $x_{0120}(r_2)$. We claim that the nontrivial coefficients contribute zero to the integral. Indeed, to show that, it is enough to prove that the integral

$$\int\limits_{(F\backslash \mathbf{A})^3} \theta^U(x_{0120}(r_2)x_{0121}(r_1)x_{0122}(r))\psi(\beta r_2+r)dr_1dr_2dr$$

is zero for all $\beta\in F^*$. It follows from Proposition 1, that this integral is zero if the integral

$$\int\limits_{(F\backslash \mathbf{A})^3} \theta_6\left[\begin{pmatrix} 1 & & & r_1 & r \\ & 1 & & r_2 & r_1 \\ & & I_2 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}\right]\psi(\beta r_2+r)dr_1dr_2dr$$

is zero for all choice of data. Here, θ_6 is a vector in the space of Θ_6. This representation was introduced right before Proposition 2, and it follows from Proposition 1 that it is a minimal representation for $\widetilde{Sp}_6(\mathbf{A})$. It follows from [G1] that the above Fourier coefficient is associated with the unipotent orbit (2^21^2). Hence it is zero for all choice of data.

Thus (49) is equal to

$$\int\limits_{(F\backslash \mathbf{A})^3} \sum_{\delta_i\in F} \theta^U(x_{0120}(r_2)x_{0121}(r_1)x_{0122}(r)w[123]x_{0120}(\delta_1)x_{1120}(\delta_2)(1,g))\psi(r)drdr_1dr_2$$

Using commutation relations and Proposition 4, one can check that as a function of g, this integral is left invariant under $x_{0100}(m_1)x_{1100}(m_2)$ for all $m_i\in\mathbf{A}$. Thus we get zero by the cuspidality of $\widetilde{\pi}$. From this we deduce that $I''=0$.

Next we consider I'. Expand the integral along $U(B_3)/U_2$ with points in $F\backslash\mathbf{A}$. Here $U(B_3)=U_{\alpha_1,\alpha_2,\alpha_3}$ is the unipotent radical of the maximal parabolic subgroup of F_4 whose Levi part is $GSpin_7$. If $x_\alpha(r)\in U(B_3)$ but not in U_2 then α is a short root. This means that if we consider a nonzero Fourier coefficient in this expansion, we get as inner integration, the Fourier coefficient which corresponds to the unipotent orbit $\widetilde{A_1}$. This Fourier coefficient is zero by the minimality of Θ. Thus we are left with the constant term. That is, I' is equal to

$$\int\limits_{SL_3(F)\backslash SL_3(\mathbf{A})} \widetilde{\varphi}(g)\theta^{U(B_3)}((1,g))dg$$

Let L_1 denote the unipotent subgroup of $Spin_7$ generated by $< x_{0120}(r); x_{1120}(r); x_{1220}(r) >$. We expand the above integral along the group $L_1(F)\backslash L_1(\mathbf{A})$. The group $SL_3(F)$, embedded

as above, acts on this expansion with two orbits. Thus I' is equal to

$$\int\limits_{SL_3(F)\backslash SL_3(\mathbf{A})} \widetilde{\varphi}(g)\theta^{U(B_3)L_1}((1,g))dg + \int\limits_{S(2)(F)\backslash SL_3(\mathbf{A})} \widetilde{\varphi}(g)\theta^{U(B_3)L_1,\psi}((1,g))dg \tag{50}$$

where

$$\theta^{U(B_3)L_1,\psi}((1,g)) = \int\limits_{(F\backslash\mathbf{A})^3} \theta^{U(B_3)}(x_{0120}(r_1)x_{1120}(r_2)x_{1220}(r_3)(1,g))\psi(r_1)dr_i$$

Let L_2 denote the group generated by $< L_1, x_{0010}(r); x_{0110}(r); x_{1110}(r) >$. In the first summand of (50) we expand the integral along L_2/L_1 with point in $F\backslash\mathbf{A}$. The group $SL_3(F)$ acts on this expansion with two orbit. The nontrivial orbit contributes the integral

$$\int\limits_{S(1)(F)\backslash SL_3(\mathbf{A})} \int\limits_{(F\backslash\mathbf{A})^3} \widetilde{\varphi}(g)\theta^{U(B_3)L_1}(x_{0010}(r_1)x_{0110}(r_2)x_{1110}(r_3)(1,g))\psi(r_1)dr_i dg$$

Since (0010) is a short root, then after a suitable conjugation, we obtain as inner integration, a Fourier coefficient which corresponds to the unipotent orbit $\widetilde{A}_1$. Thus we get zero by the minimality of Θ. The contribution of the constant term is the integral

$$\int\limits_{SL_3(F)\backslash SL_3(\mathbf{A})} \widetilde{\varphi}(g)\theta^{U(B_3)L_2}((1,g))dg$$

To show that it is zero, let $E(g,s)$ denote the Eisenstein series of $GL_3(\mathbf{A})$ associated with the induced representation $Ind_{L(\mathbf{A})}^{GL_3(\mathbf{A})}\delta_L^s$. Here L is the maximal parabolic subgroup of GL_3 whose Levi part is $GL_2 \times GL_1$. Since the identity is the residue of this Eisenstein series, then to prove that the above integral is nonzero, it is enough to prove that the integral

$$\int\limits_{SL_3(F)\backslash SL_3(\mathbf{A})} \widetilde{\varphi}(g)\theta^{U(B_3)L_2}((1,g))E(g,s)dg \tag{51}$$

is zero for $\mathrm{Re}(s)$ large. Unfolding the Eisenstein series we obtain

$$\int\limits_{S(1)(F)\backslash SL_3(\mathbf{A})} \widetilde{\varphi}(g)\theta^{U(B_3)L_2}((1,g))f(g,s)dg$$

Expand along the unipotent group $\{x_{0100}(m_2)x_{1100}(m_3)\}$. Notice that this group is the unipotent radical of $S(1)$. The group GL_2, which is the Levi part of $S(1)$ acts on this unipotent group with two orbits. The trivial one contributes zero by the cuspidality of $\widetilde{\pi}$. Thus we obtain

$$\int\limits_{T(F)N(F)\backslash SL_3(\mathbf{A})} \int\limits_{(F\backslash\mathbf{A})^2} \widetilde{\varphi}(g)\theta^{U(B_3)L_2}(x_{0100}(m_2)x_{1100}(m_3)(1,g))\psi(m_2)f(g,s)dm_i dg$$

Here N is the maximal unipotent subgroup of SL_3, and T is a one dimensional torus. We further expand along $\{x_{1000}(m_1)\}$. The trivial orbit contributes zero by cuspidality of $\widetilde{\pi}$. The nontrivial orbit contributes the integral

$$\int\limits_{(F\backslash \mathbf{A})^3} \theta^{U(B_3)L_2}(x_{1000}(m_1)x_{0100}(m_2)x_{1100}(m_3)(1,g))\psi(\gamma m_1 + m_2)dm_i$$

as inner integration. Here $\gamma \in F^*$. Applying Proposition 1 with $R = P_{\alpha_1,\alpha_2,\alpha_4}$ this integral is zero.

As for the second summand of (50), we expand the integral along the unipotent group $\{x_{1000}(r)x_{1100}(r)\}$. The group $GL_2(F)$ in $S(2)(F)$ acts on this expansion with two orbits. The orbit which corresponds to the trivial character contributes zero by the cuspidality of $\widetilde{\varphi}$. The nontrivial orbit contributes zero using Proposition 1 with $R = P_{\alpha_1,\alpha_2,\alpha_3}$. Thus $I' = 0$. This completes the proof of the cuspidality of the lift.

To show that the lift is always nonzero, we shall compute the Whittaker model of the lift. In other words, we shall compute the integral

$$W_f(h) = \int\limits_{(F\backslash \mathbf{A})^3} f(x_{0001}(r_1)x_{1231}(r_2)x_{1232}(r_3)h)\psi(r_1 + r_2)dr_i$$

We shall denote this unipotent group by V, and the above character by ψ_V. Thus we need to compute the integral

$$\int\limits_{SL_3(F)\backslash SL_3(\mathbf{A})} \int\limits_{V(F)\backslash V(\mathbf{A})} \widetilde{\varphi}(g)\theta((vh,g))\psi_V(v)dvdg$$

Following the same expansions as in the proof of the cuspidality, we obtain that all terms contribute zero except the integral

$$\int\limits_{S(1)(F)\backslash SL_3(\mathbf{A})} \int\limits_{V(F)\backslash V(\mathbf{A})} \widetilde{\varphi}(g) \sum_{\delta_i \in F} \theta^{U_2,\psi}(w[123]x_{0120}(\delta_1)x_{1120}(\delta_2)(vh,g))\psi_V(v)dvdg$$

where

$$\theta^{U_2,\psi}(m) = \int\limits_{U_2(F)\backslash U_2(\mathbf{A})} \theta(u_2 m)\psi_{U_2}(u_2)du_2$$

The group U_2 and the character ψ_{U_2} were defined in the beginning of the proof of the Proposition.

The group $SL_2(F)$ generated by $< x_{\pm(1000)}(r) >$ acts on the set $\{x_{0120}(\delta_1)x_{1120}(\delta_2) : \delta_i \in F\}$ with two orbits. First, we claim that the contribution from the trivial orbit is zero. Indeed, as explained in the proof of the cuspidality, we have

$$\theta^{U_2,\psi}(m) = \int\limits_{F\backslash A} \theta^{U_2,\psi}(x_{1111}(r)m)dr$$

This follows from the fact that (1111) is a short root, and if we expand the integral along the unipotent group $\{x_{1111}(r)\}$, then by Theorem 1, all the nontrivial Fourier coefficients will contribute zero. This means that the function $h \mapsto \theta^{U_2,\psi}(w[123](h,g))$ is left invariant by $x_{0001}(r)$ for all $r \in \mathbf{A}$. Since ψ_V is nontrivial on $x_{0001}(r)$ we get zero contribution. Thus we are left with the nontrivial orbit. Hence, we obtain

$$W_f(h) = \int\limits_{N(F)\backslash SL_3(\mathbf{A})} \int\limits_{V(F)\backslash V(\mathbf{A})} \widetilde{\varphi}(g)\theta^{U_2,\psi}(w[123]x_{1120}(1)(vh,g))\psi_V(v)dvdg$$

Here N is the maximal unipotent subgroup of SL_3.

Next, as in the proof of the cuspidality, we expand the above integral along the group U/Z with points in $F\backslash\mathbf{A}$. As in the cuspidality part, the nontrivial orbit contributes zero. Thus only the constant term contributes. Conjugating v to the left, $W_f(h)$ is equal to

$$\int\limits_{N(F)\backslash SL_3(\mathbf{A})} \widetilde{\varphi}(g) \int\limits_{(F\backslash\mathbf{A})^3} \theta^U(l(r_1,r_2,r)w[123]x_{1120}(1)(h,g))\psi(r_1+r_2+r)dr_idrdvdg \tag{52}$$

where $l(r_1,r_2,r) = x_{0111}(r_1)x_{0121}(r_2)x_{0122}(r)$. Denote

$$L(g) = \int\limits_{(F\backslash\mathbf{A})^3} \theta^U(l(r_1,r_2,r)w[123]x_{1120}(1)(h,g))\psi(r_1+r_2+r)drdr_i$$

Then, conjugating from left to right, and changing variables, we obtain

$$L(x_{1000}(m_1)x_{0100}(m_2)x_{1100}(m_3)g) =$$

$$= \int\limits_{(F\backslash\mathbf{A})^3} \theta^U(l(r_1,r_2,r)x_{0100}(m_1)x_{0120}(m_2)w[123]x_{1120}(1)(h,g))\psi(r_1+r_2+r)dr_idr$$

From Proposition 1, it follows that the function $\theta^U(m)$, when restricted to $\widetilde{Sp}_6$, is the the minimal representation Θ_6. (See before Proposition 2). Consider the integral

$$\int\limits_{(F\backslash\mathbf{A})^3} \theta^U(l(r_1,r_2,r)x_{0100}(m_1)x_{0120}(m_2))\psi(r_1+r_2+r)drdv$$

Notice that $l(r_1,r_2,r)x_{0100}(m_1)x_{0120}(m_2)$ is in Sp_6. Therefore, we can use Proposition 7. More precisely, we use the expansion (42), where to avoid confusion we shall write $U(GL_3)$ in expansion (42) instead of U. The first summand in the expansion is the constant term along $U(GL_3)$. When plugging it into the above integral we get zero because of the character $\psi(r)$. The second summand in (42) contributes

$$\int\limits_{(F\backslash\mathbf{A})^3} \sum_{\gamma\in L(GL_3)(F)\backslash GL_3(F)} \sum_{\epsilon\in\{\pm1\}\backslash F^*} \theta^{UU(GL_3),\psi}(h(\epsilon)\gamma l(r_1,r_2,r)m)\psi(r_1+r_2+r)drdv$$

where we denoted $m = x_{0100}(m_1)x_{0120}(m_2)$. We also view the matrices $h(\epsilon)$ and γ as elements in F_4 via of the embedding of Sp_6 inside F_4. The quotient $L(GL_3)(F)\backslash GL_3(F)$ is the union of the three cells

$$e;\quad \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \delta_1 \\ & & 1 \end{pmatrix};\quad \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}\begin{pmatrix} 1 & \delta_1 & \delta_2 \\ & 1 & \\ & & 1 \end{pmatrix} \tag{53}$$

Here $\delta_1, \delta_2 \in F$. It is not hard to check that the first two cells contribute zero. Indeed, this follows from the conjugation of $l(r_1, r_2, r)$ to the left across $h(\epsilon)\gamma$. As for the big cell, conjugating $l(r_1, r_2, r)$ to the left we obtain $\int \psi((\epsilon^2 - 1)r)dr$, $\int \psi((\delta_2 - 1)r_1)dr_1$ and $\int \psi(\delta_1 - 1)r_2)dr_2$ as inner integrations. Here all variables are integrated over $F\backslash\mathbf{A}$. Hence, the above integral is equal to

$$\theta^{UU(GL_3),\psi}(w[34]x_{0001}(1)x_{0011}(1)x_{0100}(m_1)x_{0120}(m_2)) =$$

$$= \psi(\frac{1}{2}(m_1 + m_2))\theta^{UU(GL_3),\psi}(w[34]x_{0001}(1)x_{0011}(1))$$

where the last equality follows from the conjugation of the m to the left, taking into an account the commutation relations in F_4.

Returning to integral (52), factoring the integration over N, we obtain

$$W_f(h) = \int\limits_{N(\mathbf{A})\backslash SL_3(\mathbf{A})} W_{\widetilde{\varphi}}(g)\theta^{UU(GL_3),\psi}(w[34]x_{0001}(1)x_{0011}(1)w[123]x_{1120}(1)(h,g))dg$$

where $W_{\widetilde{\varphi}}(g)$ is the Whittaker coefficient of the function $\widetilde{\varphi}(g)$. Using a similar argument as in [Ga-S], we deduce that $W_f(h)$ is nonzero for some choice of data, if and only if $W_{\widetilde{\varphi}}(g)$ is nonzero for some choice of data. Thus the lift is always nonzero. This completes the proof of the Proposition. □

3.1.2. **From GL_3 to $\widetilde{SL}_3$.** For this lifting we consider the following embedding of (SL_3, SL_3) inside F_4. The first copy is generated by $< x_{\pm(0001)}(r_1), x_{\pm(0010)}(r_2), x_{\pm(0011)}(r_3) >$ and the second copy is generated by $< x_{\pm(1000)}(r_1), x_{\pm(1342)}(r_2), x_{\pm(2342)}(r_3) >$. As in the previous subsection, the first copy of SL_3 splits under the cover of F_4, and the second one does not.

Let π denote a cuspidal representation of $GL_3(\mathbf{A})$. We consider the space of functions

$$\widetilde{f}(h) = \int\limits_{SL_3(F)\backslash SL_3(\mathbf{A})} \varphi(g)\theta((h,g))dg \tag{54}$$

Here φ is a vector in the space of π and $(h,g) \in (\widetilde{SL}_3(\mathbf{A}), SL_3(\mathbf{A}))$ embedded in $\widetilde{F}_4(\mathbf{A})$ as above. The function $\widetilde{f}(h)$ defines an automorphic function of $\widetilde{SL}_3(\mathbf{A})$. As we vary the data in integral (54), we obtain an automorphic representation of $\widetilde{SL}_3(\mathbf{A})$ which we shall denote by $\widetilde{\sigma}(\pi)$. First we prove

Proposition 10. *The representation $\widetilde{\sigma}(\pi)$ is a cuspidal representation of $\widetilde{SL}_3(\mathbf{A})$.*

Proof. To prove cuspidality, we have to show that the integrals

$$I=\int\limits_{V(F)\backslash V(\mathbf{A})}\widetilde{f}(vh)dv$$

are zero for all choice of data, where V is any maximal unipotent subgroup of SL_3. Up to conjugation there are two such unipotent radicals. They are given by $V_1=\{x_{1000}(r_1)x_{2342}(r_2)\}$ and $V_2=\{x_{1342}(r_1)x_{2342}(r_2)\}$. The Weyl element $w[234232]$ conjugates V_1 to V_2 and fixes the group $SL_3=< x_{\pm(0001)}(r_1),x_{\pm(0010)}(r_2),x_{\pm(0011)}(r_3) >$. Hence, to prove the cuspidality of $\widetilde{\sigma}(\pi)$, it is enough to show that the constant term of $\widetilde{f}(h)$ along $V=V_2$, is zero for all choice of data.

Let $U=U_{\alpha_2,\alpha_3,\alpha_4}$. It center was denoted by Z. Thus $Z=\{x_{2342}(r)\}\subset V$. We have

$$I=\int\limits_{SL_3(F)\backslash SL_3(\mathbf{A})}\int\limits_{Z(\mathbf{A})V(F)\backslash V(\mathbf{A})}\varphi(g)\theta^Z((h,g))dg$$

It follows from Proposition 3, that I is equal to

$$\int\limits_{SL_3(F)\backslash SL_3(\mathbf{A})}\varphi(g)\theta^U(g)dg+$$

$$\int\limits_{SL_3(F)\backslash SL_3(\mathbf{A})}\varphi(g)\int\limits_{Z(\mathbf{A})V(F)\backslash V(\mathbf{A})}\sum_{\gamma\in Q(F)\backslash Sp_6(F)}\sum_{\epsilon\in F^*}\theta^{U,\psi}(h_2(\epsilon)\gamma(v,g))dvdg$$

where Q was defined in Proposition 3. Denote the first summand by I' and the second summand by I''. From Proposition 1, it follows that I' is equal to

$$\int\limits_{SL_3(F)\backslash SL_3(\mathbf{A})}\varphi(g)\theta_6(g)dg$$

Here SL_3 is embedded in Sp_6 in the Levi part of the GL_3 parabolic subgroup, and θ_6 is a vector in the space of the representation Θ_6. (See before Proposition 2). To the above integral we apply the expansion (42) where we write $U(GL_3)$ instead of U. The first term of the expansion contributes zero to I'. Indeed, it follows from Proposition 6 that as a function of $GL_3(\mathbf{A})$, the function $\theta_6^{U(GL_3)}(g)$ is one dimensional. Hence, we obtain the integral $\int\varphi(g)dg$ as inner integration. Here g is integrated over $SL_3(F)\backslash SL_3(\mathbf{A})$. By the cuspidality of π we get zero. The second term in (42) contributes

$$\int\limits_{L'(GL_3)(F)\backslash SL_3(\mathbf{A})}\varphi(g)\theta_6^{U(GL_3),\psi}(g)dg$$

where $L'(GL_3) = L(GL_3) \cap GL_3$. Notice that $L'(GL_3)$ contains a unipotent radical of the group SL_3. Factoring this unipotent radical, and using (43), we obtain zero by the cuspidality of π. Thus $I' = 0$.

To compute I'' we consider the double coset space $Q(F)\backslash Sp_6(F)/Q(F)$. This space contains four representatives which we can choose as $e, w[2], w[232], w[232432]$. For $1 \le i \le 4$, we denote by I_i the contribution to I'' from each of the above four representatives. We start with I_1. It is equal to

$$\int\limits_{SL_3(F)\backslash SL_3(\mathbf{A})} \varphi(g) \int\limits_{Z(\mathbf{A})V(F)\backslash V(\mathbf{A})} \sum_{\epsilon \in F^*} \theta^{U,\psi}(h_2(\epsilon)(v,g))dvdg$$

From Proposition 4 it follows that for all $g \in SL_3(\mathbf{A})$ we have

$$\theta^{U,\psi}(h_2(\epsilon)(v,g)) = \theta^{U,\psi}(h_2(\epsilon)(v,1))$$

Thus we obtain the integral $\int\limits_{SL_3(F)\backslash SL_3(\mathbf{A})} \varphi(g)dg$ as inner integration. This is clearly zero, and hence $I_1 = 0$. Next, the integral I_2 is equal to

$$\int\limits_{SL_3(F)\backslash SL_3(\mathbf{A})} \varphi(g) \int\limits_{Z(\mathbf{A})V(F)\backslash V(\mathbf{A})} \sum_{\gamma \in S(4)(F)\backslash SL_3(F)} \sum_{\delta \in F, \epsilon \in F^*} \theta^{U,\psi}(h_2(\epsilon)w[2]x_{0100}(\delta)\gamma(v,g))dvdg$$

Here $S(4)$ is the maximal parabolic subgroup of SL_3 whose Levi part is GL_2 which contains the group $SL_2 =< \pm(0001) >$. This integral is equal to

$$\int\limits_{S(4)(F)\backslash SL_3(\mathbf{A})} \varphi(g) \int\limits_{Z(\mathbf{A})V(F)\backslash V(\mathbf{A})} \sum_{\delta \in F, \epsilon \in F^*} \theta^{U,\psi}(h_2(\epsilon)w[2]x_{0100}(\delta)(v,g))dvdg$$

Let $L = \{x_{0010}(l_1)x_{0011}(l_2)\}$ denote the unipotent radical of $S(4)$. Conjugating $l \in L$ to the left, using Proposition 4, we have $\theta^{U,\psi}(h_2(\epsilon)w[2]x_{0100}(\delta)(v,lg)) = \theta^{U,\psi}(h_2(\epsilon)w[2]x_{0100}(\delta)(v,g))$ for all $l \in L(\mathbf{A})$. Thus we obtain the integral $\int\limits_{L(F)\backslash L(\mathbf{A})} \varphi(lg)dl$ as inner integration. By the cuspidality of π this integral is zero. Hence $I_2 = 0$. For I_3 we obtain

$$\int\limits_{S(3)(F)\backslash SL_3(\mathbf{A})} \varphi(g) \int\limits_{Z(\mathbf{A})V(F)\backslash V(\mathbf{A})} \sum_{\delta_i \in F, \epsilon \in F^*} \theta^{U,\psi}(h_2(\epsilon)w[232]x_{0100}(\delta_1)x_{0110}(\delta_2)x_{0120}(\delta_3)(v,g))dvdg$$

where $S(3)$ is the maximal parabolic subgroup of SL_3 whose Levi part contains the group $SL_2 =< x_{\pm(0010)}(r) >$. Denote by L its unipotent radical. Thus $L = \{x_{0001}(l_1)x_{0011}(l_2)\}$. Arguing as in the case of I_2, we get zero by the cuspidality of π. Finally, I_4 is equal to

$$\int\limits_{SL_3(F)\backslash SL_3(\mathbf{A})} \varphi(g) \int\limits_{Z(\mathbf{A})V(F)\backslash V(\mathbf{A})} \sum_{\delta_i \in F} \sum_{\epsilon \in F^*} \theta^{U,\psi}(h_2(\epsilon)w[232432]y(\delta_1,\ldots,\delta_6)(v,g))dvdg \tag{55}$$

Here

$$y(\delta_1,\ldots,\delta_6) = x_{0100}(\delta_1)x_{0110}(\delta_2)x_{0111}(\delta_3)x_{0120}(\delta_4)x_{0121}(\delta_5)x_{0122}(\delta_6)$$

We have $V/Z = \{x_{1342}(r)\}$. Hence, using commutation relations

$$h_2(\epsilon)w[232432]y(\delta_1,\ldots,\delta_6)x_{1342}(r) = x_{1000}(\epsilon r)h_2(\epsilon)w[232432]y(\delta_1,\ldots,\delta_6)$$

Changing variables in U, we obtain $\int\limits_{F\backslash\mathbf{A}} \psi(\epsilon r)dr$ as inner integration. Since $\epsilon \in F^*$, this integral, and hence I_4, are both zero. This completes the proof of the Proposition. $\square$

3.1.3. **On the Nonvanishing of the Lift.** It follows from Proposition 9, that the lift from $\widetilde{GL}_3(\mathbf{A})$ to $SL_3(\mathbf{A})$ is always nonzero. In this subsection we will determine a condition on a cuspidal representation π defined on $GL_3(\mathbf{A})$ so that the lift to a cuspidal representation of $\widetilde{SL}_3(\mathbf{A})$ is nonzero. In other words, we want to find a condition on π such that the representation $\widetilde{\sigma}(\pi)$ is nonzero. This is equivalent to find a condition on π such that integral (54) is nonzero for some choice of data. From Proposition 10 it follows that $\widetilde{\sigma}(\pi)$ is a cuspidal representation. This means that $\widetilde{\sigma}(\pi)$ is nonzero if and only if it is generic. Thus we need to prove that there is a $\beta \in (F^*)^3\backslash F^*$ such that the integral

$$W_{\widetilde{f},\beta}(h) = \int\limits_{(F\backslash\mathbf{A})^3} \widetilde{f}(x_{1000}(r_1)x_{1342}(r_2)x_{2342}(r_3)h)\psi(\beta r_1 + r_2)dr_i$$

is not zero for some choice of data.

Let $\beta \in (F^*)^3\backslash F^*$. For $\mu_1,\mu_2,\mu_3 \in (F^*)^2\backslash F^*$ such that $\mu_1\mu_2\mu_3 = \beta$, consider the matrix

$$J(\mu_1,\mu_2,\mu_3) = \begin{pmatrix} & & \mu_1 \\ & \mu_2 & \\ \mu_3 & & \end{pmatrix}$$

We shall denote by $SO_3^{\mu_1,\mu_2,\mu_3}$ the orthogonal group which preserves the form given by $J(\mu_1,\mu_2,\mu_3)$.

Our result is

Proposition 11. *Suppose that the representation $\widetilde{\sigma}(\pi)$ is nonzero. Then there exists numbers μ_1,μ_2,μ_3 and β as above with $\mu_1\mu_2\mu_3 = \beta$, such that the integral*

(56)
$$\int\limits_{SO_3^{\mu_1,\mu_2,\mu_3}(F)\backslash SO_3^{\mu_1,\mu_2,\mu_3}(\mathbf{A})} \varphi(mg)dm$$

is nonzero for some choice of data.

Proof. Let $L = \{x_{1000}(r_1)x_{1342}(r_2)x_{2342}(r_3)\}$ denote the maximal unipotent subgroup of SL_3. Denote $\psi_{L,\beta}(l) = \psi(\beta r_1 + r_2)$. Thus

$$W_{\widetilde{f},\beta}(h) = \int\limits_{L(F)\backslash L(\mathbf{A})} \widetilde{f}(lh)\psi_{L,\beta}(l)dl$$

We begin the proof as in the proof of Proposition 10. Arguing as in that proof, we can show that the contribution given by I' and by I_1, I_2 and I_3 are all zero. From this we deduce that $W_{\widetilde{f},\beta}(h)$ is equal to

$$\int\limits_{SL_3(F)\backslash SL_3(\mathbf{A})} \varphi(g) \int\limits_{Z(\mathbf{A})L(F)\backslash L(\mathbf{A})} \sum_{\delta_i\in F}\sum_{\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[232432]y(\delta_1,\ldots,\delta_6)(l,g))\psi_{L,\beta}(l)dldg$$

where $y(\delta_1,\ldots,\delta_6)$ is defined as in (55). Recall that L contains the group $\{x_{1342}(l_2)\}$. Since

$$h_2(\epsilon)w[232432]y(\delta_1,\ldots,\delta_6)x_{1342}(l_2) = x_{1000}(\epsilon l_2)h_2(\epsilon)w[232432]y(\delta_1,\ldots,\delta_6)$$

we obtain the integral $\int\limits_{F\backslash \mathbf{A}} \psi((\epsilon-1)l_2)dl_2$ as inner integration. Thus only the summand with $\epsilon = 1$ contribute to the above integral. The group $SL_3(F)$ acts on the set $y(\delta_1,\ldots,\delta_6)$ via the symmetric square representation. As representatives for the various orbits, we may choose the set $x_{0100}(\mu_1)x_{0120}(\mu_2)x_{0122}(\mu_3)$ where $\mu_i\in (F^*)^2\backslash F$. We have

$$w[232432]x_{0100}(\mu_1)x_{0120}(\mu_2)x_{0122}(\mu_3)x_{1000}(l_1) =$$

$$x_{1000}(\mu_1\mu_2\mu_3 l_1)u'w[232432]x_{0100}(\mu_1)x_{0120}(\mu_2)x_{0122}(\mu_3)$$

Here $u'\in U$ such that $\psi_U(u') = 1$. Changing variables in U we obtain $\int\limits_{F\backslash \mathbf{A}} \psi((\mu_1\mu_2\mu_3 - \beta)l_1)dl_1$ as inner integration. Thus $\mu_1\mu_2\mu_3 = \beta$. In particular, all $\mu_i\neq 0$. Given such μ_i, the stabilizer of $x_{0100}(\mu_1)x_{0120}(\mu_2)x_{0122}(\mu_3)$ inside $SL_3(F)$ is given by the orthogonal group $SO_3^{\mu_1,\mu_2,\mu_3}(F)$. Thus we proved that $W_{\widetilde{f},\beta}(h)$ is equal to

$$\int\limits_{SO_3^{\mu_1,\mu_2,\mu_3}(\mathbf{A})\backslash SL_3(\mathbf{A})} \sum_{\mu_i\in (F^*)^2\backslash F^*,\mu_1\mu_2\mu_3=\beta} \varphi^{SO_3^{\mu_1,\mu_2,\mu_3}}(g)\times$$

$$\theta^{U,\psi}(w[232432]x_{0100}(\mu_1)x_{0120}(\mu_2)x_{0122}(\mu_3)(1,g)dg$$

where

$$\varphi^{SO_3^{\mu_1,\mu_2,\mu_3}}(g) = \int\limits_{SO_3^{\mu_1,\mu_2,\mu_3}(F)\backslash SO_3^{\mu_1,\mu_2,\mu_3}(\mathbf{A})} \varphi(mg)dm$$

From this the Proposition follows. □

3.2. **The Commuting pair** $(SL_2\times SL_2, Sp_4)$**.** Let $G = SL_2\times SL_2$. We embed this group inside F_4 as $H =< x_{\pm(0100)}(r); x_{\pm(0120)}(r) >$. The embedding of the group Sp_4 inside F_4 is given by

$$Sp_4 =< x_{\pm(1110)}(r); x_{\pm(0122)}(r); x_{\pm(1232)}(r); x_{\pm(2342)}(r) >$$

It thus follows that both groups do not split under the covering of F_4.

3.2.1. **From $\widetilde{SL}_2 \times \widetilde{SL}_2$ to $\widetilde{Sp}_4$.** Let $\widetilde{\pi} = \widetilde{\pi}_1 \otimes \widetilde{\pi}_2$ denote a cuspidal representation of the group $\widetilde{G}(\mathbf{A})$ where $\widetilde{\pi}_i$ are cuspidal representations of $\widetilde{SL}_2(\mathbf{A})$. Let $\widetilde{\sigma}(\widetilde{\pi})$ denote the representation of $\widetilde{Sp}_4(\mathbf{A})$ generated by all automorphic functions defined by

$$\widetilde{f}(h) = \int\limits_{G(F)\backslash G(\mathbf{A})} \widetilde{\varphi}(g)\theta((h,g))dg$$

Here $\widetilde{\varphi}$ is a function in the space of $\widetilde{\pi}$. We start with

Proposition 12. *With the above notations, suppose that $\widetilde{\pi}_1 \neq \widetilde{\pi}_2$. Then the representation $\widetilde{\sigma}(\widetilde{\pi})$ defines a cuspidal representation of $\widetilde{Sp}_4(\mathbf{A})$. Suppose further that both cuspidal representations $\widetilde{\pi}_i$, have a $\psi^{-\beta}$ Whittaker coefficient for some $\beta \in F^*$. That is, suppose that for $i = 1, 2$*

$$\int\limits_{F\backslash \mathbf{A}} \widetilde{\varphi}_i \left[\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \right] \psi(-\beta x)dx$$

is not zero for some choice of functions $\widetilde{\varphi}_i \in \widetilde{\pi}$. Then the representation $\widetilde{\sigma}(\widetilde{\pi})$ is generic.

Proof. Let $V_1 = \{x_{0122}(r_1)x_{1232}(r_2)x_{2342}(r_3)\}$ and $V_2 = \{x_{1110}(r_1)x_{1232}(r_2)x_{2342}(r_3)\}$ denote the two unipotent radicals of the two maximal parabolic subgroups of Sp_4. We need to prove that for $i = 1, 2$ the integrals

$$I = \int\limits_{V_i(F)\backslash V_i(\mathbf{A})} \widetilde{f}(vh)dv$$

are zero for all choice of data. Since both unipotent radicals contain the group Z, we can use Proposition 3 to deduce that I is equal to

$$\int\limits_{G(F)\backslash G(\mathbf{A})} \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \widetilde{\varphi}(g)\theta^U((v,g))dvdg +$$

$$\int\limits_{G(F)\backslash G(\mathbf{A})} \widetilde{\varphi}(g) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\gamma \in Q(F)\backslash Sp_6(F)} \sum_{\epsilon \in F^*} \theta^{U,\psi}(h_2(\epsilon)\gamma(v,g))dvdg$$

Denote the first summand by I' and the second by I''. Applying Proposition 1, to prove that I' is zero, it is enough to prove that

$$I_1' = \int\limits_{G(F)\backslash G(\mathbf{A})} \widetilde{\varphi}(g)\theta_6(g)dg \tag{57}$$

is zero for all choice of data. Here θ_6 is a vector in the representation Θ_6 which was defined right before Proposition 2. The embedding of G inside Sp_6 is given by $(g_1, g_2) \mapsto \mathrm{diag}(g_1, g_2, g_1^*)$. Here, for $i = 1, 2$ we have $g_i \in SL_2$. Expand the above integral along the

abelian unipotent subgroup

$$L = \left\{ \begin{pmatrix} I_2 & & X \\ & I_2 & \\ & & I_2 \end{pmatrix} ; \quad X = \begin{pmatrix} r & y \\ z & r \end{pmatrix} \right\}$$

Since Θ_6 is a minimal representation, we obtain

$$I_1' = \int\limits_{G(F)\backslash G(\mathbf{A})} \widetilde{\varphi}(g)\theta_6^L(g)dg + \int\limits_{(N_1(F)\times SL_2(F)\backslash G(\mathbf{A})} \widetilde{\varphi}(g)\theta_6^{L,\psi}(g)dg \tag{58}$$

In the first summand on the right hand side (58) we notice that $U(GL_2 \times SL_2)/L$ is an abelian group. The group $U(GL_2 \times SL_2)$ was defined at the beginning of subsection 2.6. Expanding along this quotient, it follows from the fact that Θ_6 is a minimal representation, that

$$\int\limits_{G(F)\backslash G(\mathbf{A})} \widetilde{\varphi}(g)\theta_6^L(g)dg = \int\limits_{G(F)\backslash G(\mathbf{A})} \widetilde{\varphi}(g)\theta_6^{U(GL_2\times SL_2)}(g)dg$$

From Proposition 6, and from the cuspidality of $\widetilde{\pi}$, it follows that this last integral is zero. Next consider the second summand on the right hand side of (58). In that term N_1 is the unipotent radical of SL_2 embedded in Sp_6 as $n \mapsto \text{diag}(n, I_2, n^{-1})$, and

$$\theta_6^{L,\psi}(g) = \int\limits_{L(F)\backslash L(\mathbf{A})} \theta_6(lg)\psi_L(l)dl$$

Here $\psi_L(l) = \psi(z)$ where we use the identification of L with the matrices X as was described above. We claim that the function $\theta_6^{L,\psi}(g)$ is left invariant under $N_1(\mathbf{A})$. Indeed, expanding along the group $N_1(F)\backslash N_1(\mathbf{A})$ one can show that all terms which corresponds to the non-trivial characters of the expansions, contribute zero. This follows from the fact that Θ_6 is a minimal representation. Hence $\theta_6^{L,\psi}(g) = \theta_6^{L,\psi}(ng)$ for all $n \in N_1(\mathbf{A})$. Using that in the second summand on the right hand side of (58), it follows from the cuspidality of $\widetilde{\pi}$ that it is zero. Hence $I_1' = 0$ which implies that $I' = 0$.

To compute I'' we first consider the space of double cosets $Q(F)\backslash Sp_6(F)/P(F)$ where P is the maximal parabolic subgroup of Sp_6 whose Levi part contains Sp_4. This space has two representatives which we can choose to be e and $w[234]$. Let I_1 denote the contribution to I'' from e, and I_2 the contribution from $w[234]$. Thus, I_1 is equal to

$$\int\limits_{G(F)\backslash G(\mathbf{A})} \widetilde{\varphi}(g) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\gamma\in S(3)(F)\backslash Sp_4(F)} \sum_{\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)\gamma(v,g))dvdg$$

Here $S(3)$ is the maximal parabolic subgroup of Sp_4 whose Levi part contains the SL_2 generated by $< x_{\pm(0010)}(r) >$. To proceed, we need to consider the space of double cosets

$S(3)(F)\backslash Sp_4(F)/G(F)$. This space contains two representatives which we choose to be e and $w[23]x_{0010}(1)$. The first representative contributes to I_1 the term

$$\int\limits_{B_G(F)\backslash G(\mathbf{A})} \widetilde{\varphi}(g) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)(v,g))dvdg$$

where B_G is the Borel subgroup of G. Using Proposition 4, the function $\theta^{U,\psi}(h_2(\epsilon)(v,ng))$ is invariant under $n \in N_G(\mathbf{A})$ where N_G is the maximal unipotent subgroup of G. Thus, we get zero by cuspidality.

As for the second representative, $w[23]x_{0010}(1)$, it contributes to I_1 the term

$$\int\limits_{SL_2^\Delta(F)\backslash G(\mathbf{A})} \widetilde{\varphi}(g) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[23]x_{0010}(1)(v,g))dvdg$$

Here SL_2^Δ is the group SL_2 embedded diagonally inside the group G. Using Proposition 4 we obtain $\int\limits_{SL_2^\Delta(F)\backslash SL_2^\Delta(\mathbf{A})} \widetilde{\varphi}(mg)dm$ as inner integration. By our assumption that $\widetilde{\pi}_1 \neq \widetilde{\pi}_2$, this integral is zero. Thus $I_1 = 0$.

Next, we compute I_2 which is equal to

$$\int\limits_{G(F)\backslash G(\mathbf{A})} \widetilde{\varphi}(g) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\gamma\in S(3)(F)\backslash Sp_4(F)} \sum_{\delta_i\in F,\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[234]y(\delta_1,\delta_2,\delta_3)\gamma(v,g))dvdg$$

where $y(\delta_1,\delta_2,\delta_3) = x_{0001}(\delta_1)x_{0011}(\delta_2)x_{0122}(\delta_3)$. As with I_1 we take e and $w[23]x_{0010}(1)$ for the two representatives of $S(3)(F)\backslash Sp_4(F)/G(F)$. We denote by I_{21} the contribution to I_2 from the representative e, and by I_{22} the contribution from $w[23]x_{0010}(1)$. We start with I_{22}. Since the stabilizer is SL_2^Δ, then I_{22} is equal to

$$\int\limits_{SL_2^\Delta(F)\backslash G(\mathbf{A})} \widetilde{\varphi}(g) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\delta_i\in F,\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[23423]y_1(\delta_1,\delta_2,\delta_3)x_{0010}(1)(v,g))dvdg$$

where $y_1(\delta_1,\delta_2,\delta_3) = x_{0011}(\delta_1)x_{0121}(\delta_2)x_{0122}(\delta_3)$. The unipotent element $x_{1232}(r)$ is in V_i for $i = 1,2$. We have

$$h_2(\epsilon)w[23423]y_1(\delta_1,\delta_2,\delta_3)x_{0010}(1)x_{1232}(r) = x_{1000}(\epsilon r)u'h_2(\epsilon)w[23423]y_1(\delta_1,\delta_2,\delta_3)x_{0010}(1)$$

Here $u' \in U$ is such that $\psi_U(u') = 1$. Using the left invariant properties of $\theta^{U,\psi}$, we obtain $\int\limits_{F\backslash\mathbf{A}} \psi(\epsilon r)dr$, which is clearly zero. Thus $I_{22} = 0$.

Finally, we need to consider I_{21}, which is equal to

$$\int\limits_{B_G(F)\backslash G(\mathbf{A})} \widetilde{\varphi}(g) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\delta_i\in F,\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[234]y(\delta_1,\delta_2,\delta_3)(v,g))dvdg$$

where $y(\delta_1,\delta_2,\delta_3)$ and B_G were defined above. We consider separately the cases for V_1 and V_2.

Starting with V_1, we notice that $x_{0122}(r)$ is a unipotent element in V_1. In the above integral, for $i=1$, we collapse summation and integration to obtain

$$\int\limits_{B_G(F)\backslash G(\mathbf{A})} \widetilde{\varphi}(g) \int\limits_{\mathbf{A}} \sum_{\delta_i\in F,\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[234]y(\delta_1,\delta_2,r)(1,g))drdg$$

By commutation relations, change of variables, and using Proposition 4, we obtain

$$\int\limits_{\mathbf{A}} \sum_{\delta_i\in F,\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[234]y(\delta_1,\delta_2,r)(1,x_{0100}(l)g))dr =$$

$$\int\limits_{\mathbf{A}} \sum_{\delta_i\in F,\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[234]y(\delta_1,\delta_2,r)(1,g))dr$$

for all $l\in\mathbf{A}$. Thus we get zero by the cuspidality of $\widetilde{\pi}$.

Next we consider the integral I_{21} when $V=V_2$. This time, the unipotent element $x_{1110}(r)$ is inside V_2. We have

$$h_2(\epsilon)w[234]y(\delta_1,\delta_2,\delta_3)x_{1110}(r) = x_{1000}(\epsilon\delta_1\delta_2 r)u'h_2(\epsilon)w[234]y(\delta_1,\delta_2,\delta_3)$$

where $u'\in U$ such that $\psi_U(u')=1$. Thus we obtain $\int\limits_{F\backslash\mathbf{A}} \psi(\epsilon^{-1}\delta_1\delta_2 r)dr$ as inner integration. Hence $\delta_1\delta_2=0$. From this we deduce that I_{21} is equal to

$$\int\limits_{B_G(F)\backslash G(\mathbf{A})} \widetilde{\varphi}(g) \sum_{\delta_i\in F,\delta_1\delta_2=0,\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[234]y(\delta_1,\delta_2,\delta_3)(1,g))drdg$$

If $\delta_1=0$, then for all $r\in\mathbf{A}$, using Proposition 4,

$$\theta^{U,\psi}(h_2(\epsilon)w[234]y(0,\delta_2,\delta_3)(1,x_{0120}(r)g)) = \theta^{U,\psi}(h_2(\epsilon)w[234]y(0,\delta_2,\delta_3)(1,g))$$

and if $\delta_2=0$, then for all $r\in\mathbf{A}$, using again Proposition 4,

$$\theta^{U,\psi}(h_2(\epsilon)w[234]y(\delta_1,0,\delta_3)(1,x_{0100}(r)g)) = \theta^{U,\psi}(h_2(\epsilon)w[234]y(\delta_1,0,\delta_3)(1,g))$$

Since $\{x_{0100}(r)\}$ and $\{x_{0120}(r)\}$ are the two maximal unipotent radicals of the group G, it follows that $I_{21}=0$ by the cuspidality of $\widetilde{\pi}$. This completes the cuspidality part of the Proposition.

To prove that the image of the lift is generic, we need to compute the integral

$$W_\beta(h) = \int\limits_{(F\backslash\mathbf{A})^4} \widetilde{f}(x_{1110}(r_1)x_{0122}(r_2)x_{1232}(r_3)x_{2342}(r_4)h)\psi(r_1+\beta r_2)dr_i$$

Here $\beta\in(F^*)^2\backslash F^*$. Denoting the maximal unipotent of Sp_4 by V, and the above character by $\psi_{V,\beta}$, we have to prove that the integral

$$W_\beta(h) = \int\limits_{G(F)\backslash G(\mathbf{A})} \int\limits_{V(F)\backslash V(\mathbf{A})} \widetilde{\varphi}(g)\theta((v,g))\psi_{V,\beta}(v)dvdg$$

is not zero for some choice of data. Performing the same expansions as in the proof of the cuspidality part, we obtain that all integrals except the one that corresponds to I_{21} vanish. In other words, $W_\beta(h)$ is equal to

$$\int\limits_{B_G(F)\backslash G(\mathbf{A})} \widetilde{\varphi}(g) \int\limits_{Z(\mathbf{A})V(F)\backslash V(\mathbf{A})} \sum_{\delta_i\in F,\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[234]y(\delta_1,\delta_2,\delta_3)(v,g))\psi_{V,\beta}(v)dvdg$$

As in the computation of I_{21} for the unipotent radical V_1, we collapse summation with integration. As in the computation of I_{21} for the unipotent radical V_2, we conjugate $x_{1110}(r_1)$ from right to left and we obtain that $\epsilon^{-1}\delta_1\delta_2 = 1$. Thus, $W_\beta(h)$ is equal to

$$\int\limits_{B_G(F)\backslash G(\mathbf{A})} \widetilde{\varphi}(g) \int\limits_{\mathbf{A}} \sum_{\delta_1,\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[234]y(\delta_1,\epsilon\delta_1^{-1},r)(1,g))\psi(\beta r)drdg$$

The maximal torus of G is given by $T_G = \{h(1,a,b,1) \ : \ a,b\neq 0\}$. We have the identity

$$h_2(\epsilon)w[234]x_{0001}(\delta)x_{0011}(\epsilon\delta^{-1})h(1,\epsilon^{-1},\delta^{-1},1) = h_2(\epsilon)h(1,\epsilon^{-1},\epsilon^{-1},\delta^{-1})w[234]x_{0001}(1)x_{0011}(1)$$

Using this identity, we can collapse summation and integration. Hence the above integral is equal to

$$\int\limits_{N_G(F)\backslash G(\mathbf{A})} \widetilde{\varphi}(g) \int\limits_{\mathbf{A}} \theta^{U,\psi}(w[234]y(1,1,r)(1,g))\psi(\beta r)drdg$$

where N_G is the maximal unipotent subgroup of G. In other words $N_G = \{x_{0100}(r_1)x_{0120}(r_2)\}$. Factoring the integration over N_G we obtain the identity

$$W_\beta(h) = \int\limits_{N_G(\mathbf{A})\backslash G(\mathbf{A})} W_{\widetilde{\varphi},\beta}(g) \int\limits_{\mathbf{A}} \theta^{U,\psi}(w[234]y(1,1,r)(1,g))\psi(\beta r)drdg$$

Here

$$W_{\widetilde{\varphi},\beta}(g) = \int\limits_{F\backslash\mathbf{A}} \widetilde{\varphi_1}\begin{pmatrix}1 & x\\ & 1\end{pmatrix} g_1)\psi(-\beta x)dx \int\limits_{F\backslash\mathbf{A}} \widetilde{\varphi_2}\begin{pmatrix}1 & y\\ & 1\end{pmatrix} g_1)\psi(-\beta y)dy$$

where $\widetilde{\varphi} = \widetilde{\varphi_1}\otimes\widetilde{\varphi_2}$ and $g = (g_1, g_2)$.

From this it is clear that if the lift is non-zero then $W_{\widetilde{\varphi},\beta}(g)$ is not zero. Using a similar argument as in [Ga-S], it follows that the converse is also true. Namely, if $W_{\widetilde{\varphi},\beta}(g)$ is not zero then the lift to $\widetilde{Sp}_4$ is not zero. □

3.2.2. **From $\widetilde{Sp}_4$ to $\widetilde{SL}_2\times\widetilde{SL}_2$.** To study this lifting, we consider a different embedding of the two groups. We embed the group Sp_4 as the Levi part of the corresponding parabolic subgroup of F_4. In other words $Sp_4 =< x_{\pm(0100)}(r), x_{\pm(0010)}(r) >$. The group $G = SL_2\times SL_2$ is generated by $< x_{\pm(0122)}(r); x_{\pm(2342)}(r) >$.

Let $\widetilde{\pi}$ denote a cuspidal representation of $\widetilde{Sp}_4(\mathbf{A})$. We shall denote by $\widetilde{\sigma}(\widetilde{\pi})$ the automorphic representation of $G(\mathbf{A})$ generated by all functions of the form

$$\widetilde{f}(g)=\int\limits_{Sp_4(F)\backslash Sp_4(\mathbf{A})}\widetilde{\varphi}(h)\theta((g,h))dh$$

Here $\widetilde{\varphi}$ is a vector in the space of $\widetilde{\pi}$. We start with

Proposition 13. *The representation $\widetilde{\sigma}(\widetilde{\pi})$ defines a cuspidal representation of $G(\mathbf{A})$.*

Proof. Since the two unipotent radicals which correspond to the two maximal parabolic subgroups of G, are conjugated one to the other inside F_4, it is enough to prove that the integral

$$\int\limits_{Sp_4(F)\backslash Sp_4(\mathbf{A})}\int\limits_{F\backslash\mathbf{A}}\widetilde{\varphi}(h)\theta((x_{2342}(r),h))drdh$$

is zero for all choice of data. From Proposition 3, this integral is equal to

$$\int\limits_{Sp_4(F)\backslash Sp_4(\mathbf{A})}\widetilde{\varphi}(h)\theta^U((1,h))dh+\int\limits_{Sp_4(F)\backslash Sp_4(\mathbf{A})}\widetilde{\varphi}(h)\sum_{\gamma\in Q(F)\backslash Sp_6(F)}\sum_{\epsilon\in F^*}\theta^{U,\psi}(h_2(\epsilon)(1,h))dh$$

Denote the first summand by I_1, and the second by I'. From Proposition 1, it follows that the first summand is zero. Indeed, it is zero if the integral

$$\int\limits_{Sp_4(F)\backslash Sp_4(\mathbf{A})}\widetilde{\varphi}(h)\theta_6(h)dh$$

is zero for all choice of data. To prove that we expand along $Z(F)\backslash Z(\mathbf{A})$ where Z was defined right before Proposition 8. Thus, the above integral is equal to

$$\int\limits_{Sp_4(F)\backslash Sp_4(\mathbf{A})}\widetilde{\varphi}(h)\theta_6^Z(h)dh+\sum_{\beta\in F^*}\int\limits_{Sp_4(F)\backslash Sp_4(\mathbf{A})}\widetilde{\varphi}(h)\theta_6^{Z,\psi_\beta}(h)dh \tag{59}$$

In the first term, we use the fact that $U(GL_1\times Sp_4)/Z$ is an abelian subgroup. See beginning of subsection 2.6 for notations. From the fact that Θ_6 is a minimal representation, we deduce that

$$\int\limits_{Sp_4(F)\backslash Sp_4(\mathbf{A})}\widetilde{\varphi}(h)\theta_6^Z(h)dh=\int\limits_{Sp_4(F)\backslash Sp_4(\mathbf{A})}\widetilde{\varphi}(h)\theta_6^{U(GL_1\times Sp_4)}(h)dh$$

Arguing in a similar way as in integral (58) we deduce that the above integral is zero for all choice of data. The notations of the second summand of (59) are as in Proposition 8, and it follows from that Proposition that each term in the second summand of (59) is zero. Indeed, from Proposition 8 it follows that each term is equal to

$$\int\limits_{Sp_4(F)\backslash Sp_4(\mathbf{A})}\widetilde{\varphi}(h)\theta_{Sp_4}^{\phi,\psi_\beta}(h)dh$$

By cuspidality, this integral is zero. Thus $I_1 = 0$.

Next consider the integral I'. Let P denote the maximal parabolic subgroup of Sp_6 whose Levi part contains Sp_4. The space $Q(F)\backslash Sp_6(F)/P(F)$ contains two elements which we can choose to be e and $w[234]$. The contribution to I' from the identity element is

$$\int\limits_{S(3)(F)\backslash Sp_4(\mathbf{A})} \widetilde{\varphi}(h) \sum_{\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)(1,h))dh$$

where $S(3)$ is the maximal parabolic subgroup of Sp_4 whose Levi part contains the group generated by $< x_{\pm(0010)}(r) >$. Denote the unipotent radical of $S(3)$ by $N(3)$. Then it follows from Proposition 4 that the integral $\int\limits_{N(3)(F)\backslash N(3)(\mathbf{A})} \widetilde{\varphi}(nh)dn$ is an inner integration to the above integral. By the cuspidality of $\widetilde{\pi}$ this integral is zero.

The second representative contributes to I' the integral

$$\int\limits_{S(3)(F)\backslash Sp_4(\mathbf{A})} \widetilde{\varphi}(h) \sum_{\delta_i\in F,\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[234]y(\delta_1,\delta_2,\delta_3)(1,h))dh$$

where $y(\delta_1,\delta_2,\delta_3) = x_{0001}(\delta_1)x_{0011}(\delta_2)x_{0122}(\delta_3)$. If $\delta_1 = \delta_2 = 0$ then as in the previous representative, we factor the subgroup $N(3)$ to get zero contribution. Otherwise, the group $SL_2(F)$ which is generated by $< x_{\pm(0010)}(r) >$ acts on the set $\{x_{0001}(\delta_1)x_{0011}(\delta_2) : (\delta_1,\delta_2) \neq (0,0)\}$ with one orbit. Thus the above integral is equal to

$$\int\limits_{T(F)N(F)\backslash Sp_4(\mathbf{A})} \widetilde{\varphi}(h) \sum_{\delta_3\in F,\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[234]y(0,1,\delta_3)(1,h))dh$$

where N is the maximal unipotent subgroup of Sp_4 and T is a one dimensional torus. Let $S(2)$ denote the maximal parabolic subgroup of Sp_4 whose Levi part is $GL_1\times SL_2$. Let $N(2)$ denote it's unipotent radical. Thus $N(2) = \{x_{0100}(r_1)x_{0110}(r_2)x_{0120}(r_3)\}$. Using commutation relations, it follows from Proposition 4 that the function

$$h \mapsto \theta^{U,\psi}(h_2(\epsilon)w[234]y(0,1,\delta_3)(1,h))$$

is left invariant under $N(2)(\mathbf{A})$. Thus we get zero by the cuspidality of $\widetilde{\pi}$. Hence $I' = 0$ and the lift is cuspidal. □

Next we consider the question of the nonvanishing of the lift. To do that we need to find conditions so that the integral

$$W_{\widetilde{f},\beta}(g) = \int\limits_{Sp_4(F)\backslash Sp_4(\mathbf{A})} \int\limits_{(F\backslash\mathbf{A})^2} \widetilde{\varphi}(h)\theta((x_{0122}(r_1)x_{2342}(r_2)g,h))\psi(\beta r_1 + r_2)dr_i dh$$

will not be zero for some choice of data. Here $\beta \in F^*$.

For $\delta \in F^*$, let SO_4^δ denote the stabilizer insider SO_5 of a vector of length δ. We have

Proposition 14. *The representation $\widetilde{\sigma}(\widetilde{\pi})$ is nonzero if and only there exists $\beta \in F^*$ such that the integral*

$$\int\limits_{SO_4^\beta(F)\backslash SO_4^\beta(\mathbf{A})} \widetilde{\varphi}(m)\theta_{Sp_4}^{\phi,\psi}(m)dm \tag{60}$$

is not zero for some choice of data.

Proof. We compute $W_{\widetilde{f},\beta}(g)$. Using Proposition 5, the integral $W_{\widetilde{f},\beta}(g)$ is not zero for some choice of data, if and only if the integral

$$\int\limits_{Sp_4(F)\backslash Sp_4(\mathbf{A})} \int\limits_{F\backslash\mathbf{A}} \widetilde{\varphi}(h)\theta_{Sp_{14}}^{\phi',\psi}(\varpi_3(x_{0122}(r_1)h))\psi(\beta r_1)dr_1dh$$

is not zero for some choice of data. The group we integrate over is a subgroup of $SL_2 \times Sp_4$ embedded inside Sp_6 in the obvious way. Thus, from the restriction of ϖ_3 to this subgroup it follows from the well known factorization of the theta function, that the above integral is equal to

$$\int\limits_{Sp_4(F)\backslash Sp_4(\mathbf{A})} \int\limits_{F\backslash\mathbf{A}} \widetilde{\varphi}(h)\theta_{Sp_4}^{\phi,\psi}(h)\theta_{Sp_{10}}^{\phi_1,\psi}((\varpi_2(h),x(r_2))\psi(\beta r_2)dr_2dh$$

Here $\varpi_2(g)$ is the degree five representation of Sp_4. Also, by $(\varpi_2(g),x(r_2))$ we mean the embedding of these groups inside the commuting pair $SO_5 \times SL_2$ inside Sp_{10}. Unfolding the theta function of Sp_{10}, we obtain only one orbit, corresponding to vectors of length β. The stabilizer is the group we denoted by SO_4^β. Thus, $W_{\widetilde{f},\beta}(g)$ is equal to

$$\int\limits_{SO_4^\beta(F)\backslash Sp_4(\mathbf{A})} \widetilde{\varphi}(h)\theta_{Sp_4}^{\phi,\psi}(h)\phi_1(l(\beta)h)dh$$

where $l(\beta)$ is a vector in F^5 whose length is β. Factoring the measures, integral (60) appears as an inner integration. From this the Proposition follows. □

3.3. **The Commuting Pair** (SL_2, SL_4)**.** In this subsection we will study the lifting from automorphic representations defined on $SL_2(\mathbf{A})$ to automorphic representations defined on $\widetilde{SL}_4(\mathbf{A})$, and its inverse map. We start with:

3.3.1. **From** GL_2 **to** $\widetilde{SL}_4$**.** We consider the following embedding of (SL_2, SL_4) inside the group F_4. The group SL_2 is generated by $< x_{\pm(0001)}(r) >$. The group SL_4 is the group generated by

$$< x_{\pm(1000)}(r); x_{\pm(0100)}(r); x_{\pm(1242)}(r); x_{\pm(1100)}(r); x_{\pm(1342)}(r); x_{\pm(2342)}(r) >$$

Since SL_2 is generated by unipotent elements which correspond to short roots, this copy splits under the double cover of F_4.

Let π denote an irreducible cuspidal representation of $GL_2(\mathbf{A})$. We shall denote by $\widetilde{\sigma}(\pi)$ the automorphic representation of $\widetilde{SL}_4$ spanned by all automorphic functions

$$\widetilde{f}(h)=\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})}\varphi(g)\theta((h,g))dg$$

Here $h\in\widetilde{SL}_4(\mathbf{A})$. We shall denote by $L(\pi,s)$ the standard L function associated with π. We prove

Proposition 15. *Suppose that π is an irreducible cuspidal representation of $GL_2(\mathbf{A})$ such that $L(\pi,1/2)=0$. Then, $\widetilde{\sigma}(\pi)$ defines a nonzero cuspidal representation of $\widetilde{SL}_4(\mathbf{A})$.*

Proof. We start with the cuspidality condition. The group SL_4 has three maximal parabolic subgroups. Their unipotent radicals are given by $V_1=\{x_{1242}(r_1)x_{1342}(r_2)x_{2342}(r_3)\}$, $V_2=\{x_{0100}(r_1)x_{1100}(r_2)x_{1342}(r_3)x_{2342}(r_4)\}$ and $V_3=\{x_{1000}(r_1)x_{1100}(r_2)x_{2342}(r_3)\}$. The Weyl element $w[3243423]$ conjugates V_3 to V_1 and fixes the group SL_2 generated by $< x_{\pm(0001)}(r) >$. Hence it is enough to prove that for $i=1,2$, the integral

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})}\ \int\limits_{V_i(F)\backslash V_i(\mathbf{A})}\varphi(g)\theta((v,g))dvdg \tag{61}$$

is zero for all choice of data. Both unipotent subgroups V_i contains the group $Z=\{x_{2342}(r)\}$. Hence, using Proposition 3, integral (61) is equal to the sum

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})}\ \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})}\varphi(g)\theta^U((v,g))dvdg+$$

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})}\varphi(g)\int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})}\ \sum_{\gamma\in Q(F)\backslash Sp_6(F)}\ \sum_{\epsilon\in F^*}\theta^{U,\psi}(h_2(\epsilon)\gamma(v,g))dvdg$$

Denote the first integral by I' and the second one by I''. From Proposition 1, it follows that the integral

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})}\varphi(g)\theta_6(d(g))dg$$

is an inner integration to integral I'. Here, for all $g\in SL_2(\mathbf{A})$, we set $d(g)=diag(g,I_2,g^*)$, and θ_6 is a vector in the space of the representation Θ_6, which was defined right before Proposition 2. To prove this integral is zero we proceed exactly as in the proof that integral (57) is zero for all choice of data. Indeed, as can be seen the proof of that integral only uses one copy of SL_2, the one which we embedded here as $\{d(g):g\in SL_2\}$. Hence $I'=0$.

Next we compute I''. As in the proof of Proposition 10, for $1\le j\le 4$, we denote by I_j the contribution to I'' from each of the double coset representatives of $Q(F)\backslash Sp_6(F)/Q(F)$,

which we choose as $e, w[2], w[232]$ and $w[232432]$. The integral I_1 is equal to

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})}\ \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \varphi(g)\sum_{\epsilon\in F^*}\theta^{U,\psi}(h_2(\epsilon)(v,g))dvdg$$

Using Proposition 4 we obtain $\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})}\varphi(g)dg$ as inner integration. Thus $I_1=0$. Next, I_4 is equal to

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})}\ \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \varphi(g)\sum_{\delta_i\in F \epsilon\in F^*}\theta^{U,\psi}(h_2(\epsilon)w[232432]m(\delta_i)(v,g))dvdg$$

Here

$$m(\delta_i)=x_{0100}(\delta_1)x_{0110}(\delta_2)x_{0111}(\delta_3)x_{0120}(\delta_4)x_{0121}(\delta_5)x_{0122}(\delta_6)$$

Notice that V_i contains the one dimensional unipotent subgroup $x_{1342}(r)$. From the identity $w[232432]m(\delta_i)x_{1342}(r)=x_{1000}(r)w[232432]m(\delta_i)$, we obtain $\int\limits_{F\backslash\mathbf{A}}\psi(\epsilon r)dr$ as inner integration. Thus $I_4=0$.

Integral I_2 is equal to

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})}\varphi(g)\int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})}\ \sum_{\gamma\in S(4)(F)\backslash SL_3(F)}\ \sum_{\delta\in F,\epsilon\in F^*}\theta^{U,\psi}(h_2(\epsilon)w[2]x_{0100}(\delta)\gamma(v,g))dvdg$$

Here $S(4)$ is the maximal parabolic subgroup of SL_3 which contains the group $< x_{\pm 0001}(r) >$. The space $S(4)(F)\backslash SL_3(F)/S(4)(F)$ contains two representatives, which we can choose as e and $w[3]$. The first representative contributes zero to I_2. Indeed, it is equal to

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})}\varphi(g)\int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})}\ \sum_{\delta\in F,\epsilon\in F^*}\theta^{U,\psi}(h_2(\epsilon)w[2]x_{0100}(\delta)(v,g))dvdg$$

It follows from Proposition 4, that for all $g\in SL_2(\mathbf{A})$ we have $\theta^{U,\psi}(h_2(\epsilon)w[2]x_{0100}(\delta)(v,g))=\theta^{U,\psi}(h_2(\epsilon)w[2]x_{0100}(\delta)(v,1))$. Hence, we obtain the integral $\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})}\varphi(g)dg$ as inner integration. Thus, I_2 is equal to

$$\int\limits_{B_2(F)\backslash SL_2(\mathbf{A})}\varphi(g)\int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})}\ \sum_{\delta_i\in F,\epsilon\in F^*}\theta^{U,\psi}(h_2(\epsilon)w[23]x_{0010}(\delta_1)x_{0120}(\delta_2)(v,g))dvdg$$

where B_2 is the Borel subgroup of SL_2. From commutation relations in F_4, and using Proposition 4, we deduce that the function

$$g\mapsto\theta^{U,\psi}(h_2(\epsilon)w[23]x_{0010}(\delta_1)x_{0120}(\delta_2)(v,g))$$

is left invariant under $x_{0001}(r)$ for all $r\in\mathbf{A}$. Thus we obtain the integral $\int\limits_{F\backslash\mathbf{A}}\varphi(\begin{pmatrix}1 & x\\ & 1\end{pmatrix})dx$ as inner integration. From the cuspidality of π it follows that this last integral, and hence I_2, is zero.

Finally, we consider I_3. It is equal to

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \varphi(g) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\substack{\gamma\in S(3)(F)\backslash SL_3(F)\\ \delta_i\in F,\epsilon\in F^*}} \theta^{U,\psi}(h_2(\epsilon)w[232]y_1(\delta_1,\delta_2,\delta_3)\gamma(v,g))dvdg$$

where $y_1(\delta_1,\delta_2,\delta_3) = x_{0100}(\delta_1)x_{0110}(\delta_2)x_{0120}(\delta_3)$. Also, $S(3)$ is the maximal parabolic subgroup of SL_3 which contains the group $< x_{\pm 0010}(r) >$. The space $S(3)(F)\backslash SL_3(F)/S(4)(F)$ contains two elements which we choose as e and $w[43]$. The contribution to I_3 from e is equal to

$$\int\limits_{B_2(F)\backslash SL_2(\mathbf{A})} \varphi(g) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\delta_i\in F,\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[232]x_{0100}(\delta_1)x_{0110}(\delta_2)x_{0120}(\delta_3)(v,g))dvdg$$

As above, it follows from Proposition 4 that the function

$$g \mapsto \theta^{U,\psi}(h_2(\epsilon)w[232]y_1(\delta_1,\delta_2,\delta_3)(v,g))$$

is left invariant by $x_{0001}(r)$ for all $r \in \mathbf{A}$. Hence we get zero contribution from this term. Thus I_3 is equal to

$$(62) \qquad \int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \varphi(g) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\delta_i\in F,\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[23243]y(\delta_1,\ldots,\delta_5)(v,g))dvdg$$

Here

$$y(\delta_1,\ldots,\delta_5) = x_{0120}(\delta_1)x_{0121}(\delta_2)x_{0122}(\delta_3)x_{0010}(\delta_4)x_{0011}(\delta_5)$$

Suppose first that $i = 1$. Then $x_{1242}(r) \in V_1$. We have

$$h_2(\epsilon)w[23243]y(\delta_1,\ldots,\delta_5)x_{1242}(r) = x_{1000}(\epsilon^{-1}r)h_2(\epsilon)w[23243]y(\delta_1,\ldots,\delta_5)$$

Changing variables in U, we obtain $\int\limits_{F\backslash\mathbf{A}} \psi(\epsilon^{-1}r)dr$ as inner integration. Hence I_3 is zero in this case.

Next suppose that $i = 2$. The group $SL_2(F)$ generated by $< x_{\pm(0001)}(\mu) >$, acts on the group $\{y(0,0,0,\delta_4,\delta_5) : \delta_i \in F\}$ with two orbits.

Consider first the trivial orbit. We denote the contribution to I_3 from this term by I_{31}. Then we consider the action of the above $SL_2(F)$ on the group $\{y(\delta_1,\delta_2,\delta_3,0,0) : \delta_i \in F\}$. The action is given by the symmetric square representation. There are infinite number of orbits. First, using the cuspidality of π, the trivial orbit and the orbits which correspond to a nonzero vector with zero length, all contribute zero to the integral I_{31}. Indeed, for the trivial orbit we obtain $\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \varphi(g)dg$ as inner integration, and for the orbit which corresponds to nonzero vectors with zero length, we obtain $\int\limits_{N_2(F)\backslash N_2(\mathbf{A})} \varphi(ng)dn$ as inner integration. Here N_2 is the maximal unipotent subgroup of SL_2. Clearly both integrals are zero.

Thus we are left with the orbits which correspond to a vector of nonzero length. There are infinite number of such vectors, and the stabilizer inside $SL_2(F)$ of any such orbit, is an orthogonal group $O_2(F)$. Factoring the measure, and using Proposition 4 we obtain $\int\limits_{O_2(F)\backslash O_2(\mathbf{A})} \varphi(mg)dm$ as inner integration. The type of the orthogonal group, depends on the representative of the orbit. From [W] it follows that the vanishing of $L(\pi, 1/2)$ is equivalent to the vanishing of all the above integrals over O_2. Thus $I_{31} = 0$, and I_3 is equal to

$$\int\limits_{N_2(F)\backslash SL_2(\mathbf{A})} \varphi(g) \int\limits_{Z(\mathbf{A})V_2(F)\backslash V_2(\mathbf{A})} \sum_{\delta_i\in F, \epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[23243]y(\delta_1, \delta_2, \delta_3, 0, 1)(v, g))dvdg$$

where N_2 is the maximal unipotent subgroup of SL_2. The unipotent elements $x_{0100}(r_1)$ and $x_{1100}(r_2)$ are inside V_2. Using commutation relations we have,

$$h_2(\epsilon)w[23243]y(\delta_1, \delta_2, \delta_3, 0, 1)x_{1100}(r_2) = vux_{1000}(\epsilon^{-1}\delta_1 r_2)h_2(\epsilon)w[23243]y(\delta_1, \delta_2, \delta_3, 0, 1)$$

where v is an element in the stabilizer of ψ_U and $u \in U$ such that $\psi_U(u) = 1$. Thus, changing variables in U, we obtain $\int\limits_{F\backslash\mathbf{A}} \psi(\epsilon^{-1}\delta_1 r_2)dr_2$ as inner integration. Hence, we may assume that $\delta_1 = 0$. Next, using commutation relations we obtain

$$h_2(\epsilon)w[23243]y(0, \delta_2, \delta_3, 0, 1)x_{0100}(r_1) = vuh_2(\epsilon)w[23243]y(0, \delta_2, \delta_3 + r_1, 0, 1)$$

where u and v are as above. Collapsing summation with integration, I_3 is equal to

$$\int\limits_{N_2(F)\backslash SL_2(\mathbf{A})} \varphi(g) \int\limits_{\mathbf{A}} \sum_{\delta_2\in F, \epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[23243]y(0, \delta_2, r_1, 0, 1)(1, g))dr_1dg \tag{63}$$

Using Proposition 4, the function

$$g \mapsto \int\limits_{\mathbf{A}} \sum_{\delta_i\in F, \epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[23243]y(0, \delta_2, r_1, 0, 1)(1, g))dr_1$$

is left invariant under $x_{0001}(r)$ for all $r \in \mathbf{A}$. Thus, $I_3 = 0$ by the cuspidality of π. Hence integral (61) is zero for all unipotent radicals V_i. This completes the proof of the cuspidality of the lift.

To prove the nonvanishing of the lift, we shall compute the Whittaker function of $\widetilde{f}$, where $\widetilde{f}$ is in the space of $\widetilde{\sigma}(\pi)$. Let $\beta \in (F^*)^4\backslash F^*$. For $h \in \widetilde{SL}_4(\mathbf{A})$, denote by $W_{\widetilde{f},\beta}(h)$ the integral

$$\int\limits_{(F\backslash A)^6} \widetilde{f}(x_{1000}(r_1)x_{0100}(r_2)x_{1242}(r_3)x_{1100}(r_4)x_{1342}(r_5)x_{2342}(r_6)h)\psi(\beta r_1 + r_2 + r_3)dr_i$$

We shall denote this unipotent group by V and the above character by $\psi_{V,\beta}$. Thus, $W_{\widetilde{f},\beta}(h)$ is equal to

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \int\limits_{V(F)\backslash V(\mathbf{A})} \varphi(g)\theta((vh, g))\psi_{V,\beta}(v)dvdg$$

Following the same steps as we did in the proof of the cuspidality of $\widetilde{\sigma}(\pi)$, we obtain that all integrals, except (62), contribute zero to $W_{\widetilde{f},\beta}(h)$. Thus $W_{\widetilde{f},\beta}(h)$ is equal to

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \varphi(g) \int\limits_{Z(\mathbf{A})V(F)\backslash V(\mathbf{A})} \sum_{\delta_i\in F,\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[23243]y(\delta_1,\ldots,\delta_5)(vh,g))\psi_{V,\beta}(v)dvdg$$

Using the commutation relations as after (62), and arguing as in integral (63), we deduce that $\epsilon=1$. Continuing further as in the proof of the cuspidality, $W_{\widetilde{f},\beta}(h)$ is equal to

$$\int\limits_{N_2(F)\backslash SL_2(\mathbf{A})} \varphi(g) \int\limits_{\mathbf{A}} \int\limits_{F\backslash\mathbf{A}} \sum_{\delta_2\in F} \theta^{U,\psi}(w[23243]y(0,\delta_2,r_2,0,1)(x_{1000}(r_1),g))\psi(\beta r_1+r_2)dr_1dr_2dg$$

Next we conjugate the unipotent element $x_{1000}(r_1)$ to the left. We have

$$w[23243]y(0,\delta_2,r_2,0,1)x_{1000}(r_1)=x_{1000}(\delta_2r_1)u'w[23243]y(0,\delta_2,r_2,0,1)$$

Here $u'\in U$ is such that $\psi_U(u')=1$. Thus, we obtain the integral $\int\limits_{F\backslash\mathbf{A}}\psi((\delta_2-\beta)r_1)dr_1$ as inner integration. From this we deduce that $\delta_2=\beta$. Hence, $W_{\widetilde{f},\beta}(h)$ is equal to

$$\int\limits_{N_2(F)\backslash SL_2(\mathbf{A})} \varphi(g) \int\limits_{\mathbf{A}} \theta^{U,\psi}(w[23243]y(0,\beta,r_2,0,1)(1,g))\psi(r_2)dr_2dg$$

Using commutation relations and a change of variables, we obtain

$$\int\limits_{\mathbf{A}} \theta^{U,\psi}(w[23243]y(0,\beta,r_2,0,1)(1,x_{0001}(r)g))\psi(r_2)dr_2=$$

$$\psi(\beta r)\int\limits_{\mathbf{A}} \theta^{U,\psi}(w[23243]y(0,\beta,r_2,0,1)(1,g))\psi(r_2)dr_2$$

From this we obtain the identity

$$W_{\widetilde{f},\beta}(h)=\int\limits_{N_2(\mathbf{A})\backslash SL_2(\mathbf{A})} W_{\varphi,\beta}(g)\int\limits_{\mathbf{A}} \theta^{U,\psi}(w[23243]y(0,\beta,r_2,0,1)(1,g))\psi(r_2)dr_2dg$$

where $W_{\varphi,\beta}(g)=\int\limits_{F\backslash\mathbf{A}}\varphi(\begin{pmatrix}1 & r\\ & 1\end{pmatrix}g)\psi(\beta r)dr$. Using similar arguments as in [Ga-S] we deduce that $W_{\widetilde{f},\beta}(h)$ is nonzero for some choice of data if and only if $W_{\varphi,\beta}(g)$ is nonzero for some choice of data. Since there is always a $\beta\in F^*$ such that $W_{\varphi,\beta}(g)$ is not zero, the nonvanishing of the lift follows. $\square$

3.3.2. **From $\widetilde{SL}_4$ to SL_2.** To study this lift we consider a different embedding of the two groups. Viewing SL_4 as $Spin_6$, we embed it inside the Levi part of the maximal parabolic subgroup of F_4 whose Levi part contains $Spin_7$. Thus, the group SL_4 is generated by

$$< x_{\pm(1000)}(r); x_{\pm(0100)}(r); x_{\pm(1100)}(r); x_{\pm(0120)}(r); x_{\pm(1120)}(r); x_{\pm(1220)}(r) >$$

The group SL_2 is generated by $< x_{\pm(1232)}(r) >$. If we conjugate these groups by the Weyl element $w[3213234]$ we obtain the embedding we used in the previous subsection.

Let $\widetilde{\pi}$ denote an irreducible cuspidal representation defined on $\widetilde{SL}_4(\mathbf{A})$. We shall denote by $\sigma(\widetilde{\pi})$ the automorphic representation of $SL_2(\mathbf{A})$ spanned by all functions

$$f(g) = \int\limits_{SL_4(F)\backslash SL_4(\mathbf{A})} \widetilde{\varphi}(h)\theta((h,g))dh$$

Here $\widetilde{\varphi}$ is a vector in the space of $\widetilde{\pi}$. We start with

Proposition 16. *The representation $\sigma(\widetilde{\pi})$ is nonzero if and only if the integral*

$$\int\limits_{Sp_4(F)\backslash Sp_4(\mathbf{A})} \widetilde{\varphi}(m)\theta_{Sp_4}^{\phi,\psi}(m)dm \tag{64}$$

is nonzero for some choice of data. Here $\theta_{Sp_4}^{\phi,\psi}$ is the theta function defined on $\widetilde{Sp}_4(\mathbf{A})$.

Proof. Clearly, $\sigma(\widetilde{\pi})$ is nonzero if and only if the integral

$$W_f(g) = \int\limits_{SL_4(F)\backslash SL_4(\mathbf{A})} \int\limits_{F\backslash \mathbf{A}} \widetilde{\varphi}(h)\theta((h, x_{1232}(r)g))\psi(\beta r)drdh$$

is nonzero for some choice of data. Let U_1 denote the abelian unipotent group generated by all elements of the form

$$u_1(r_1,\ldots,r_6) = x_{0122}(r_1)x_{1122}(r_2)x_{1222}(r_3)x_{1242}(r_4)x_{1342}(r_5)x_{2342}(r_6)$$

and let $U_2 =< U_1, x_{1232}(r) >$. We expand θ along the group $U_1(F)\backslash U_1(\mathbf{A})$. The group $SL_4(F) = Spin_6(F)$ acts on this expansion with three type of orbits. The first two orbits are the ones which corresponds to the trivial orbit, and to the orbit corresponding to nonzero vectors with zero length. Plugging these two Fourier coefficients in $W_f(g)$ we obtain the integrals

$$\int\limits_{U_1(F)\backslash U_1(\mathbf{A})} \int\limits_{F\backslash \mathbf{A}} \theta(u_1(r_1,\ldots,r_6)x_{1232}(r))\psi(\epsilon r_1 + \beta r)drdr_i$$

as inner integrations. Here $\epsilon = 0$ when the orbit is the trivial one, and $\epsilon = 1$ corresponds to the other orbit. In both cases, the above Fourier coefficient corresponds to the unipotent orbit $\widetilde{A}_1$ which is greater than the minimal orbit. Hence, by Theorem 1, these Fourier coefficients are zero.

The third type of orbits corresponds to vectors of nonzero length. These contributes the Fourier coefficient

$$\int\limits_{U_1(F)\backslash U_1(\mathbf{A})} \int\limits_{F\backslash \mathbf{A}} \theta(u_1(r_1,\ldots,r_6)x_{1232}(r))\psi(r_3 + \gamma r_4 + r)drdr_i$$

where $\gamma \in F^*$. Notice that the stabilizer of this character inside $Spin_6 = SL_4$ is $Spin_5 = Sp_4$. We can identify the group $U_2(\mathbf{A})$ with $\mathbf{A}^7$. With this identification we can write the above integral as

$$\int\limits_{U_2(F)\backslash U_2(\mathbf{A})} \theta(u_2)\psi(\delta \cdot u_2)du_2$$

Here we identify u_2 with a column vector and $\delta = (0, 0, 1, 1, \gamma, 0, 0)$. With this identification $\delta \cdot u_2$ is the usual dot product. If γ is such that δ has a nonzero length, then this Fourier coefficient corresponds to the unipotent orbit $\widetilde{A}_1$, and as above, it is zero. There is one choice of γ such that the length of δ is zero. Conjugating by a suitable discrete element, this Fourier coefficient is equal to

$$\int\limits_{U_2(F)\backslash U_2(\mathbf{A})} \theta(u_2 w[123]x_{0010}(1))\psi_{U_2}(u_2)du_2$$

where ψ_{U_2} is defined as follows. For $u_2 = x_{0122}(r)u_2'$ define $\psi_{U_2}(u_2) = \psi(r)$. See subsection 2.1 for notations. From this we obtain that $W_f(g)$ is equal to

$$\int\limits_{Sp_4(F)\backslash SL_4(\mathbf{A})} \int\limits_{U_2(F)\backslash U_2(\mathbf{A})} \widetilde{\varphi}(h)\theta(u_2 w[123]x_{0010}(1)(h,1))\psi_{U_2}(u_2)du_2$$

Using Proposition 3 this integral is equal to

$$\int\limits_{Sp_4(F)\backslash SL_4(\mathbf{A})} \int\limits_{Z(\mathbf{A})U_2(F)\backslash U_2(\mathbf{A})} \widetilde{\varphi}(h)\theta^U(u_2\mu(h,1))\psi_{U_2}(u_2)du_2dh+$$

$$\int\limits_{Sp_4(F)\backslash SL_4(\mathbf{A})} \varphi(g) \int\limits_{Z(\mathbf{A})U_2(F)\backslash U_2(\mathbf{A})} \sum_{\gamma\in Q(F)\backslash Sp_6(F)} \sum_{\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)\gamma u_2\mu(h,1))\psi_{U_2}(u_2)du_2dh$$

where we denote $\mu = w[123]x_{0010}(1)$. Denote the first integral by I' and the second one by I''. We start with I''. Let P denote the maximal parabolic subgroup of Sp_6 whose Levi part contains Sp_4. The space $Q(F)\backslash Sp_6(F)/P(F)$ has two representatives which we can choose as e and $w[234]$. The first representative contributes

$$\int\limits_{S(3)(F)\backslash SL_4(\mathbf{A})} \varphi(g) \int\limits_{Z(\mathbf{A})U_2(F)\backslash U_2(\mathbf{A})} \sum_{\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)u_2\mu(h,1))\psi_{U_2}(u_2)du_2dg$$

to the integral. Here $S(3)$ is the parabolic subgroup of Sp_4 whose Levi part is GL_2. Changing variables in U and using Proposition 4, we obtain that

$$\theta^{U,\psi}(h_2(\epsilon)u_2\mu(h,1)) = \theta^{U,\psi}(h_2(\epsilon)\mu(h,1))$$

for all $u_2 \in U_2(\mathbf{A})$. Thus we obtain $\int\limits_{Z(\mathbf{A})U_2(F)\backslash U_2(\mathbf{A})} \psi_{U_2}(u_2)du_2$ as inner integration. Thus the contribution to I'' from this term is zero. The second representative contributes the integral

$$\int\limits_{S(3)(F)\backslash SL_4(\mathbf{A})} \varphi(g) \int\limits_{Z(\mathbf{A})U_2(F)\backslash U_2(\mathbf{A})} \sum_{\delta_i \in F, \epsilon \in F^*} \theta^{U,\psi}(h_2(\epsilon)w[234]y(\delta_1,\delta_2,\delta_3)u_2\mu(h,1))\psi_{U_2}(u_2)du_2dg$$

where $y(\delta_1,\delta_2,\delta_3) = x_{0001}(\delta_1)x_{0011}(\delta_2)x_{0122}(\delta_3)$. We have $h_2(\epsilon)w[234]y(\delta_1,\delta_2,\delta_3)x_{1122}(r) = x_{1000}(\epsilon^{-1}r)h_2(\epsilon)w[234]y(\delta_1,\delta_2,\delta_3)$. Hence we get $\int\limits_{F\backslash \mathbf{A}} \psi(\epsilon^{-1}r)dr$ as inner integration. This integral is zero and hence $I'' = 0$. Thus $W_f(g)$ is equal to I'. Factoring the measure, we obtain

$$\int\limits_{Sp_4(\mathbf{A})\backslash SL_4(\mathbf{A})} \int\limits_{Sp_4(F)\backslash Sp_4(\mathbf{A})} \int\limits_{Z(\mathbf{A})U_2(F)\backslash U_2(\mathbf{A})} \widetilde{\varphi}(mh)\theta^U(u_2\mu(mh,1))\psi_{U_2}(u_2)du_2dmdh$$

Arguing as in [Ga-S] we deduce that the lift is nonzero for some choice of data if and only if the integral

$$\int\limits_{Sp_4(F)\backslash Sp_4(\mathbf{A})} \int\limits_{F\backslash \mathbf{A}} \widetilde{\varphi}(m)\theta^U(x_{0122}(r)\mu(m,1))\psi(r)drdm$$

is nonzero for some choice of data. The group $\mu Sp_4\mu^{-1} =< x_{\pm(0100)}(r); x_{\pm(0010)}(r) >$. Hence, from Proposition 1 it follows that the lift is nonzero for some choice of data if and only if the integral

$$\int\limits_{Sp_4(F)\backslash Sp_4(\mathbf{A})} \int\limits_{F\backslash \mathbf{A}} \widetilde{\varphi}(m)\theta_6(x_{0122}(r)m)\psi(r)drdm$$

is not zero for some choice of data. It follows from Proposition 8 that the above integral is not zero for some choice of data, if and only if integral (64) is not zero for some choice of data. This completes the proof of the Proposition. □

Next we address the question of cuspidality of the lift. We prove

Proposition 17. *The representation $\sigma(\widetilde{\pi})$ is a cuspidal representation of $SL_2(\mathbf{A})$.*

Proof. We need to show that the integral

$$\int\limits_{SL_4(F)\backslash SL_4(\mathbf{A})} \int\limits_{F\backslash \mathbf{A}} \widetilde{\varphi}(h)\theta((h, x_{1232}(r)g))drdh \tag{65}$$

is zero for all choice of data. We expand the theta function along the group U_1 which was defined in the proof of Proposition 16. Combining this with the integration over the group $\{x_{1232}(r)\}$, integral (65) is equal to

$$\int\limits_{SL_4(F)\backslash SL_4(\mathbf{A})} \widetilde{\varphi}(h)\theta^{U_2}(u_2(h,1))du_2dh + \int\limits_{S(F)\backslash SL_4(\mathbf{A})} \int\limits_{U_2(F)\backslash U_2(\mathbf{A})} \widetilde{\varphi}(h)\theta(u_2(h,1))\psi_{U_2}(u_2)du_2dh$$

Here S is the subgroup of SL_4 defined by

$$S =< x_{\pm(0100)}(r); x_{\pm(0120)}(r); x_{(1000)}(r); x_{(1100)}(r); x_{(1120)}(r); x_{(1220)}(r) >$$

and ψ_{U_2} was defined in the proof of Proposition 16. Denote by I' the first summand and by I'' the second one. We start with I''. Using Proposition 3 it is equal to

$$\int\limits_{S(F)\backslash SL_4(\mathbf{A})} \int\limits_{Z(\mathbf{A})U_2(F)\backslash U_2(\mathbf{A})} \widetilde{\varphi}(h)\theta^U(u_2(h,1))\psi_{U_2}(u_2)du_2dh+$$

$$\int\limits_{S(F)\backslash SL_4(\mathbf{A})} \varphi(g) \int\limits_{Z(\mathbf{A})U_2(F)\backslash U_2(\mathbf{A})} \sum_{\gamma\in Q(F)\backslash Sp_6(F)} \sum_{\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)\gamma u_2(h,1))\psi_{U_2}(u_2)du_2dh$$

Arguing in a similar way as in the computation of I'' in the proof of Proposition 16, we deduce that the second summand in the above integral is zero. Indeed, using Proposition 4 we obtain the integral

$$\int\limits_{Mat_{2\times 2}(F)\backslash Mat_{2\times 2}(\mathbf{A})} \widetilde{\varphi}(\begin{pmatrix} I_2 & X \\ & I_2 \end{pmatrix})dX$$

as inner integration to the first summand. This is zero by the cuspidality of $\widetilde{\pi}$.

Let $U(B_3) = U_{\alpha_1,\alpha_2,\alpha_3}$. Then U_2 is a subgroup of $U(B_3)$. The quotient $U(B_3)/U_2$ is an eight dimensional abelian group and SL_4 acts on it as twice the standard representation. The quotient $U(B_3)/U_2$ is generated by all unipotent groups $\{x_\alpha(r)\}$ such that $\alpha = \sum_{i=1}^3 n_i\alpha_i+\alpha_4$. To compute I' we further expend it along the group $U(B_3)/U_2$ with points in $F\backslash\mathbf{A}$. By the minimality of Θ only the constant term contributes. Indeed, the nontrivial Fourier coefficients will contain, as inner integration, a Fourier coefficient which corresponds to the unipotent orbit $\widetilde{A}_1$. This follows from the fact that the length of all the above roots α is short. Thus

$$I' = \int\limits_{SL_4(F)\backslash SL_4(\mathbf{A})} \widetilde{\varphi}(h)\theta^{U(B_3)}((h,1))dh$$

To show that this last integral is zero, let $E(h,s)$ denote the Eisenstein series of $SL_4(\mathbf{A})$ which is associated with the induced representation $Ind_{R(\mathbf{A})}^{SL_4(\mathbf{A})}\delta_R^s$. Here R is a maximal parabolic subgroup of SL_4 whose Levi part is GL_3. Thus, to prove that I' is zero, it is enough to show that for $Re(s)$ large, the integral

$$\int\limits_{SL_4(F)\backslash SL_4(\mathbf{A})} \widetilde{\varphi}(h)\theta^{U(B_3)}(h)E(h,s)dh$$

is zero for all choice of data. Unfolding the Eisenstein series, and using from [PS1], the well known Whittaker expansion of $\widetilde{\varphi}$ we obtain the integral

$$\int\limits_{V(F)\backslash V(\mathbf{A})} \theta^{U(B_3)}(vh)\psi_V(v)dv$$

as inner integration. Here V is the maximal unipotent subgroup of $Spin_6 = SL_4$ and ψ_V is the Whittaker coefficient of V. The Fourier coefficient given by the above integration over V is a Fourier coefficient which is associated to a unipotent orbit of $Spin_7$ which is greater than the minimal orbit. Hence, it follows from Proposition 1 that this integral is zero. This implies that I' is zero. This completes the proof of the cuspidality of the lift. □

3.4. **The Commuting Pair (SO_3, G_2).** In this subsection we will consider the lift of automorphic representations from the group $SO_3(\mathbf{A})$ to automorphic representations of the exceptional $\widetilde{G}_2(\mathbf{A})$. We first consider

3.4.1. **From $SO_3(\mathbf{A})$ to $\widetilde{G}_2(\mathbf{A})$.** To study this lift, we consider the following embedding of the two groups. The group SO_3 is generated by $\{x_{0010}(r)x_{0001}(-r)x_{0011}(-r^2)\}$ and by $\{x_{-(0010)}(r)x_{-(0001)}(-r)x_{-(0011)}(-r^2)\}$. In other words, we embed SO_3 inside the group SL_3 generated by $< x_{\pm(0010)}(r); x_{\pm(0001)}(r) >$. With this choice, the group V, the maximal unipotent subgroup of G_2, is generated by

$$V =< x_{1000}(r); x_{0120}(r)x_{0111}(r); x_{1111}(r)x_{1120}(r); x_{1231}(r)x_{1222}(r); x_{1342}(r); x_{2342}(r) >$$

The group G_2 is generated by V and by the group generated by all unipotent elements which corresponds to the negative roots of the above six unipotent elements. With this choice of embedding, the group SO_3 splits under the double cover, but G_2 does not.

Let π denote an irreducible cuspidal representation of the group $SO_3(\mathbf{A})$. Let $\widetilde{\sigma}(\pi)$ denote the automorphic representation of $\widetilde{G}_2(\mathbf{A})$ generated by all functions

$$\widetilde{f}(h) = \int\limits_{SO_3(F)\backslash SO_3(\mathbf{A})} \varphi(g)\theta((h,g))dg$$

where $h \in \widetilde{G}_2(\mathbf{A})$ and $\varphi(g)$ is a vector in the space of π. We shall denote by $L(\pi, s)$ the standard L function attached to π. We prove

Proposition 18. *Let π be as above, and assume that $L(\pi, 1/2) = 0$. Then $\widetilde{\sigma}(\pi)$ defines a generic cuspidal representation of $\widetilde{G}_2(\mathbf{A})$.*

Proof. For $i = 1, 2$, we shall denote by V_i the two unipotent radicals of the maximal parabolic subgroups of G_2. In other words,

$$V_1 =< x_{0120}(r)x_{0111}(r); x_{1111}(r)x_{1120}(r); x_{1231}(r)x_{1222}(r); x_{1342}(r); x_{2342}(r) >$$

and

$$V_2 =< x_{1000}(r); x_{1111}(r)x_{1120}(r); x_{1231}(r)x_{1222}(r); x_{1342}(r); x_{2342}(r) >$$

To prove the cuspidality of the lift, we need to prove that for $i=1,2$ the integrals

$$\int\limits_{SO_3(F)\backslash SO_3(\mathbf{A})}\int\limits_{V_i(F)\backslash V_i(\mathbf{A})}\varphi(g)\theta((v,g))dvdg$$

are zero for all choice of data. The group $Z=\{x_{2342}(r)\}$ is a subgroup of V_i. Hence, using Proposition 3 this integral is equal to

$$\int\limits_{SO_3(F)\backslash SO_3(\mathbf{A})}\int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})}\varphi(g)\theta^U((v,g))dvdg+$$

$$\int\limits_{SO_3(F)\backslash SO_3(\mathbf{A})}\varphi(g)\int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})}\sum_{\gamma\in Q(F)\backslash Sp_6(F)}\sum_{\epsilon\in F^*}\theta^{U,\psi}(h_2(\epsilon)\gamma(v,g))dvdg$$

Denote by I_1' the first summand and by I' the second summand. To show that the first summand is zero, using Proposition 1, it is enough to show that the integral

$$\int\limits_{SO_3(F)\backslash SO_3(\mathbf{A})}\varphi(g)\theta_6((1,g))dg$$

is zero for all choice of data. Here θ_6 is a vector in the space of the representation Θ_6 which was defined right before Proposition 2. Apply Proposition 7 to this integral, by using identity (42). The contribution of the constant term gives us the integral

$$\int\limits_{SO_3(F)\backslash SO_3(\mathbf{A})}\varphi(g)\theta_6^{U(GL_3)}((1,g))dg$$

Here $U(GL_3)$ is the unipotent radical of the parabolic group $P(GL_3)$ which was defined before Proposition 6. In Proposition 7 this unipotent group was denoted by U. From Proposition 6 we obtain the integral $\int\limits_{SO_3(F)\backslash SO_3(\mathbf{A})}\varphi(g)dg$ as inner integration. This integral is clearly zero. Plugging the second summand of (42) we obtain

$$\int\limits_{SO_3(F)\backslash SO_3(\mathbf{A})}\varphi(g)\sum_{\gamma\in L_0(GL_3)(F)\backslash GL_3(F)}\theta_6^{U(GL_3),\psi}(\gamma(1,g))dg$$

where $L_0(GL_3)$ was defined right before Proposition 7. Consider the space of double cosets $L_0(GL_3)(F)\backslash GL_3(F)/SO_3(F)$. We partition the set of representatives δ into two sets. The first set has the property that $\delta^{-1}L_0(GL_3)\delta\cap SO_3$ is the maximal unipotent subgroup of SO_3. In this case, from the cuspidality of π and from equation (43) in Proposition 7, we get zero contribution. The other type of representative has the property that $\delta^{-1}L_0(GL_3)\delta\cap SO_3$ is a certain SO_2 which can be embedded in the split SO_3. Thus, applying again equation (43) in Proposition 7 we get $\int\limits_{SO_2(F)\backslash SO_2(\mathbf{A})}\varphi(g)dg$ as inner integration. From [W], we know that if $L(\pi,1/2)=0$ then this integral is zero. Thus $I_1'=0$.

Next we compute I'. The space of the double cosets $Q(F)\backslash Sp_6(F)/Q(F)$ contains four representatives which we can choose to be $e, w[2], w[232]$ and $w[232432]$. For $1 \le j \le 4$, we denote by I_j the contribution to I' from each one of the four representatives. First, integral I_1 is equal to

$$\int\limits_{SO_3(F)\backslash SO_3(\mathbf{A})} \varphi(g) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)(v,g))dvdg$$

Using Proposition 4 we obtain $\int\limits_{SO_3(F)\backslash SO_3(\mathbf{A})} \varphi(g)dg$ as inner integration. Thus $I_1 = 0$. Next, I_4 is equal to

$$\int\limits_{SO_3(F)\backslash SO_3(\mathbf{A})} \varphi(g) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\delta_i\in F,\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[232432]y(\delta_1,\ldots,\delta_6)(v,g))dvdg$$

where

$$y(\delta_1,\ldots,\delta_6) = x_{0100}(\delta_1)x_{0110}(\delta_2)x_{0111}(\delta_3)x_{0120}(\delta_4)x_{0121}(\delta_5)x_{0122}(\delta_6)$$

Since $h_2(\epsilon)w[232432]y(\delta_1,\ldots,\delta_6)x_{1342}(r) = x_{1000}(\epsilon^{-1}r)h_2(\epsilon)w[232432]y(\delta_1,\ldots,\delta_6)$ we obtain $\int\limits_{F\backslash\mathbf{A}} \psi(\epsilon^{-1}r)dr$ as inner integration. Thus $I_4 = 0$.

Integral I_2 is equal to

$$\int\limits_{SO_3(F)\backslash SO_3(\mathbf{A})} \varphi(g) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\gamma\in S(4)(F)\backslash SL_3(F)} \sum_{\delta_1\in F,\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[2]x_{0100}(\delta_1)\gamma(v,g))dvdg$$

Here $S(4)$ is the maximal parabolic subgroup of SL_3 whose Levi part contains the SL_2 generated by $< x_{\pm(0001)}(r) >$. The space $S(4)(F)\backslash SL_3(F)/SO_3(F)$ contains infinite number of orbits. As representative we can choose $e, w[3]$ and $w[34]x_{0011}(\nu)$ where $\nu \in (F^*)^2\backslash F^*$. The identity representative contributes to I_2 the term

$$\int\limits_{B(F)\backslash SO_3(\mathbf{A})} \varphi(g) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\delta_1\in F,\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[2]x_{0100}(\delta_1)(v,g))dvdg$$

where B is the Borel subgroup of SO_3. Let N denote the unipotent radical of SO_3. From Proposition 4 we deduce that the function

$$g \mapsto \theta^{U,\psi}(h_2(\epsilon)w[2]x_{0100}(\delta_1)(v,g))$$

is left invariant under all $n \in N(\mathbf{A})$. Thus we get zero by the cuspidality of π. The second representative contributes to I_2 the term

$$\int\limits_{T(F)\backslash SO_3(\mathbf{A})} \varphi(g) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\delta_1\in F,\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[23]x_{0120}(\delta_1)(v,g))dvdg$$

where T is the maximal split torus of SO_3. From Proposition 4 it follows that the function

$$g \mapsto \theta^{U,\psi}(h_2(\epsilon)w[23]x_{0120}(\delta_1)(v,g))$$

is left invariant under $T(\mathbf{A})$. Thus we obtain $\int\limits_{T(F)\backslash T(\mathbf{A})} \varphi(tg)dt$ as inner integration. Since $L(\pi, 1/2) = 0$, it follows that this last integral is zero. Thus we are left with the third family of representatives which contributes to I_2 the integral

$$\sum_{\nu\in F^*} \int\limits_{S_\nu(F)\backslash SO_3(\mathbf{A})} \varphi(g) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\delta_1\in F,\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[234]x_{0122}(\delta_1)x_{0011}(\nu)(v,g))dvdg$$

where S_ν is an orthogonal group which depends on ν. We have $x_{1111}(r)x_{1120}(r) \in V_i$. Also, we have the commutation relations

$$h_2(\epsilon)w[234]x_{0122}(\delta_1)x_{0011}(\nu)x_{1111}(r)x_{1120}(r) = x_{1000}(\nu\epsilon^{-1}r)u'h_2(\epsilon)w[234]x_{0122}(\delta_1)x_{0011}(\nu)$$

where $u' \in U$ such that $\psi_U(u') = 1$. Thus we obtain $\int\limits_{F\backslash\mathbf{A}} \psi(\nu\epsilon^{-1}r)dr$ as inner integration. Hence $I_2 = 0$.

We are left with I_3 which is equal to

$$\int\limits_{SO_3(F)\backslash SO_3(\mathbf{A})} \varphi(g) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\substack{\gamma\in S(3)(F)\backslash SL_3(F)\\ \delta_i\in F\epsilon\in F^*}} \theta^{U,\psi}(h_2(\epsilon)w[232]y_1(\delta_1,\delta_2,\delta_3)\gamma(v,g))dvdg$$

where $y_1(\delta_1,\delta_2,\delta_3) = x_{0100}(\delta_1)x_{0110}(\delta_2)x_{0120}(\delta_3)$ and $S(3)$ is the maximal parabolic subgroup of SL_3 which contains the SL_2 generated by $< x_{\pm 0010}(r) >$. As in the computations of I_2, the space $S(3)(F)\backslash SL_3(F)/SO_3(F)$ contains infinite number of orbits. As representative we can choose $e, w[4]$ and $w[43]x_{0011}(\nu)$ where $\nu \in (F^*)^2\backslash F^*$. The contribution from the identity element is

$$\int\limits_{B(F)\backslash SO_3(\mathbf{A})} \varphi(g) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\delta_i\in F,\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[232]y_1(\delta_1,\delta_2,\delta_3)(v,g))dvdg$$

The unipotent element $x_{1111}(r)x_{1120}(r) \in V_i$. We have

$$x_{0100}(\delta_1)x_{1111}(r)x_{1120}(r) = x_{1111}(r)x_{1120}(r)x_{1220}(\delta_1 r)x_{0100}(\delta_1)$$

Since $w[232]x_{1220}(\delta_1 r) = x_{1000}(\delta_1 r)w[232]$, we obtain $\int\limits_{F\backslash\mathbf{A}} \psi(\delta_1\epsilon^{-1}r)dr$ as inner integration. Thus we may assume that $\delta_1 = 0$. If $\delta_2 = 0$, then the function

$$g \mapsto \theta^{U,\psi}(h_2(\epsilon)w[232]y_1(0,0,\delta_3)(v,g))$$

is left invariant under $N(\mathbf{A})$. Thus, by cuspidality we get zero. Hence we may assume that we sum over $\delta_2 \neq 0$. The torus $T(F)$ acts transitively on the set $\{x_{0110}(\delta_2) : \delta_2 \neq 0\}$. Collapsing summation with integration, the above integral is equal to

$$\int\limits_{N(F)\backslash SO_3(\mathbf{A})} \varphi(g) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\delta_3\in F,\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[232]x_{0110}(1)x_{0120}(\delta_3)(v,g))dvdg$$

Suppose first that $i=1$. Then $x_{0111}(r)x_{0120}(r)\in V_1$. Collapsing summation with integration, we obtain as inner integration, the integral

$$\int\limits_{\mathbf{A}}\sum_{\epsilon\in F^*}\theta^{U,\psi}(h_2(\epsilon)w[232]x_{0110}(1)x_{0120}(r)(v,g))dr$$

By commutation relations, one can check that as a function of g the above integral is left invariant under $N(\mathbf{A})$. Thus we get zero by the cuspidality of π. When $i=2$, we have $x_{1000}(r)\in V_2$. We have

$$h_2(\epsilon)w[232]x_{0110}(1)x_{0120}(\delta_3)x_{1000}(r)=x_{1000}(\epsilon^{-1}r)u'h_2(\epsilon)w[232]x_{0110}(1)x_{0120}(\delta_3)$$

where $u'\in U$ is such that $\psi_U(u')=1$. Thus we get zero in this case also. Next we consider the contribution of $w[4]$ to I_3. It is equal to

$$\int\limits_{T(F)\backslash SO_3(\mathbf{A})}\varphi(g)\int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})}\sum_{\delta_i\in F,\epsilon\in F^*}\theta^{U,\psi}(h_2(\epsilon)w[2324]y_2(\delta_1,\delta_2,\delta_3)(v,g))dvdg$$

where $y_2(\delta_1,\delta_2,\delta_3)=x_{0100}(\delta_1)x_{0111}(\delta_2)x_{0122}(\delta_3)$. We have $x_{1231}(r)x_{1222}(r)\in V_i$. Since $w[2324]x_{1222}(r)=x_{1000}(r)w[2324]$ we get zero contribution in this case. Finally, the last set of representatives are

$$\sum_{\nu\in F^*}\int\limits_{S_\nu(F)\backslash SO_3(\mathbf{A})}\varphi(g)\int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})}\sum_{\delta_1\in F,\epsilon\in F^*}\theta^{U,\psi}(h_2(\epsilon)w_0y_3(\delta_1,\delta_2,\delta_3)x_{0011}(\nu)(v,g))dvdg$$

where $y_3(\delta_1,\delta_2,\delta_3)=x_{0120}(\delta_1)x_{0121}(\delta_2)x_{0122}(\delta_3)$ and $w_0=w[23243]$. As above, we use the unipotent matrix $x_{1231}(r)x_{1222}(r)$ to get zero. Thus I_3 equal to zero. This completes the proof of the cuspidality.

To prove the nonvanishing of the lift we compute its Whittaker coefficient. In other words, we compute the integral

$$W_{\widetilde{f}}(h)=\int\limits_{V(F)\backslash V(\mathbf{A})}\widetilde{f}(vh)\psi_V(v)dv$$

where ψ_V is defined as follows. For $v\in V$ write $v=x_{1000}(r_1)x_{0120}(r_2)x_{0111}(r_2)v'$. Then $\psi_V(v)=\psi(r_1+r_2)$. See subsection 2.1 for notations. Repeating the same expansions as in the proof of the cuspidality, we obtain zero contribution except from the term which corresponds to the identity representative in the computation of I_3. In other words $W_{\widetilde{f}}(h)$ is equal to

$$\int\limits_{N(F)\backslash SO_3(\mathbf{A})}\varphi(g)\int\limits_{(F\backslash\mathbf{A})^2}\sum_{\delta\in F,\epsilon\in F^*}\theta^{U,\psi}(h_2(\epsilon)w[232]x_{0110}(1)x_{0120}(\delta)(y(r_1,r_2)h,g))\psi(r_1+r_2)dr_idg$$

where $y(r_1,r_2)=x_{1000}(r_1)x_{0120}(r_2)x_{0111}(r_2)$. We have

$$h_2(\epsilon)w[232]x_{0110}(1)x_{0120}(\delta)x_{1000}(r_1)=x_{1000}(\epsilon^{-1}r_1)u'h_2(\epsilon)w[232]x_{0110}(1)x_{0120}(\delta)$$

where $u' \in U$ is such that $\psi_U(u') = 1$. Thus we get $\int\limits_{F\backslash \mathbf{A}} \psi((1-\epsilon^{-1})r)dr$ as inner integration, and hence $\epsilon = 1$. Next, collapsing summation over δ_2 with the integration over r_2, the above integral is equal to

$$\int\limits_{N(F)\backslash SO_3(\mathbf{A})} \varphi(g) \int\limits_{\mathbf{A}} \theta^{U,\psi}(w[232]x_{0110}(1)x_{0120}(r_2)(h,g))\psi(r_2)dr_2dg$$

Factoring the integration over N, we obtain the identity

$$W_{\widetilde{f}}(h) = \int\limits_{N(\mathbf{A})\backslash SO_3(\mathbf{A})} W_\varphi(g) \int\limits_{\mathbf{A}} \theta^{U,\psi}(w[232]x_{0110}(1)x_{0120}(r_2)(h,g))\psi(r_2)dr_2dg$$

where $W_\varphi(g)$ is the Whittaker coefficient attached to φ. This completes the proof of the Proposition. $\square$

3.4.2. **From $\widetilde{G_2}$ to SO_3.** To study this lift, we consider a different embedding of the commuting pair. First, we embed the group G_2 inside F_4 as the group generated by all unipotent elements $< x_{\pm(1000)}(r)x_{\pm(0010)}(r); x_{\pm(0100)}(m) >$. This embedding is the standard embedding of the group G_2 inside $Spin_7$. The group SO_3 is the group generated by $< x_{\pm(0001)}(r)x_{\pm(1231)}(-r)x_{\pm(1232)}(-r^2) >$. This embedding and the one introduced in the previous subsection are related by conjugation of the Weyl element $w[231234]$. We shall denote by V the unipotent radical subgroup of the standard Borel subgroup of G_2 embedded as above.

Let $\widetilde{\pi}$ denote a cuspidal irreducible representation of the group $\widetilde{G_2}(\mathbf{A})$. Let $\sigma(\widetilde{\pi})$ denote the automorphic representation of $SO_3(\mathbf{A})$ generated by all functions of the form

$$f(g) = \int\limits_{G_2(F)\backslash G_2(\mathbf{A})} \widetilde{\varphi}(h)\theta((h,g))dh$$

We start with

Proposition 19. *The representation $\sigma(\widetilde{\pi})$ is a cuspidal representation of $SO_3(\mathbf{A})$.*

Proof. Let $x(r) = x_{0001}(r)x_{1231}(-r)x_{1232}(-r^2)$. We need to prove that the integral

$$\int\limits_{G_2(F)\backslash G_2(\mathbf{A})} \int\limits_{F\backslash \mathbf{A}} \widetilde{\varphi}(h)\theta((h,x(r)g))drdh$$

is zero for all choice of data. We expand the integral along the group U_2. This group was defined in the beginning of the proof of Proposition 16. As explained there, there are only two orbits which contributes nonzero terms. They correspond to the constant term and to

the set of all nonzero vectors with zero length. Thus, the above integral is equal to

$$\int\limits_{G_2(F)\backslash G_2(\mathbf{A})}\int\limits_{F\backslash \mathbf{A}}\widetilde{\varphi}(h)\theta^{U_2}((h,x(r)g))drdh+\int\limits_{S(2)(F)\backslash G_2(\mathbf{A})}\int\limits_{F\backslash \mathbf{A}}\widetilde{\varphi}(h)\theta^{U_2,\psi}((h,x(r)g))drdh$$

Denote the first summand by I' and the second one by I''. In the above integral, $S(2)$ is the subgroup of G_2 generated by $< x_{\pm(0100)}(r), V >$. Also, we denoted

$$\theta^{U_2,\psi}(m)=\int\limits_{U_2(F)\backslash U_2(\mathbf{A})}\theta(u_2m)\psi_{U_2}(u_2)du_2$$

where ψ_{U_2} was defined in the proof of Proposition 16. We mention, that in the computation of I'' we used the fact that G_2 acts transitively on the set of all nonzero vectors with zero length.

We start with I''. Let $U=U_{\alpha_2,\alpha_3,\alpha_4}$. Expand the integral along the group U/Z with points in $F\backslash\mathbf{A}$. Using Proposition 3 we obtain two terms. Thus, I'' is equal to

$$\int\limits_{S(2)(F)\backslash G_2(\mathbf{A})}\int\limits_{(F\backslash \mathbf{A})^2}\widetilde{\varphi}(h)\theta^{U}(x_{0122}(r_1)(h,x(r)g))\psi(r_1)dr_1drdh+$$

$$\int\limits_{S(2)(F)\backslash G_2(\mathbf{A})}\int\limits_{F\backslash \mathbf{A}}\widetilde{\varphi}(h)\int\limits_{Z(\mathbf{A})U_2(F)\backslash U_2(\mathbf{A})}\sum_{\substack{\gamma\in Q(F)\backslash Sp_6(F)\\ \epsilon\in F^*}}\theta^{U,\psi}(h_2(\epsilon)\gamma u_2(h,x(r)g))\psi_{U_2}(u_2)du_2drdh$$

Arguing as in the proof of Proposition 9, it is not hard to check that the second summand is zero. As for the first one, after conjugation and changing variables in U, we obtain

$$\int\limits_{S(2)(F)\backslash G_2(\mathbf{A})}\int\limits_{(F\backslash \mathbf{A})^2}\widetilde{\varphi}(h)\theta^{U}(x_{0121}(r)x_{0122}(r_1)(h,g))\psi(r_1)dr_1drdh$$

The function

$$L(h)=\int\limits_{(F\backslash \mathbf{A})^2}\theta^{U}(x_{0121}(r)x_{0122}(r_1)(h,g))\psi(r_1)dr_1dr$$

is left invariant by the unipotent radical $V(2)$ of the maximal parabolic subgroup $S(2)$. Indeed, we have

$$V(2)=\{x_{1000}(m_1)x_{0010}(m_1)x_{1100}(m_2)x_{0110}(m_2)x_{1110}(m_3)x_{0120}(m_3)x_{1120}(m_4)x_{1220}(m_5)\}$$

Changing variables in U the above integral is equal to

$$\int\limits_{(F\backslash \mathbf{A})^2}\theta^{U}(x_{0121}(r)x_{0122}(r_1)y(m_1,m_2,m_3))\psi(r_1)dr_1dr$$

where $y(m_1,m_2,m_3)=x_{0010}(m_1)x_{0110}(m_2)x_{0120}(m_3)$. It follows from Proposition 1 that θ^U is the representation Θ_6 defined on $\widetilde{Sp}_6$ right before Proposition 2. Thus, the above integral

is equal to

$$\int\limits_{(F\backslash \mathbf{A})^2} \theta_6\left[\begin{pmatrix}1&&&&r&r_1\\&1&&&&r\\&&1&&&\\&&&1&&\\&&&&1&\\&&&&&1\end{pmatrix}\begin{pmatrix}1&&&&&\\&1&m_1&m_2&m_3&\\&&1&&m_2&\\&&&1&-m_1&\\&&&&1&\\&&&&&1\end{pmatrix}\right]\psi(r_1)drdr_1$$

Denote the right most matrix by m. Plug in the above integral expansion (42) in Proposition 7. The first term contributes zero since we obtain $\int\limits_{F\backslash \mathbf{A}} \psi(r_1)dr_1$ as inner integration. The second summand in expansion (42), when plugged inside the above integral can be written as a union of cells given by (53). It is not hard to check that the first two cells contribute zero. The last cell contributes

$$\sum_{\delta_i\in F}\int\limits_{(F\backslash \mathbf{A})^2} \theta_6^{U(GL_3),\psi}\left[w\begin{pmatrix}1&\delta_1&\delta_2&&&\\&1&&&&\\&&1&&&\\&&&1&&-\delta_2\\&&&&1&-\delta_1\\&&&&&1\end{pmatrix}\begin{pmatrix}1&&&&r&r_1\\&1&&&&r\\&&1&&&\\&&&1&&\\&&&&1&\\&&&&&1\end{pmatrix}m\right]\psi(r_1)drdr_1$$

Here, in equation (42) we wrote $U(GL_3)$ instead of U. Also,

$$w=\begin{pmatrix}w_0&\\&w_0^*\end{pmatrix};\qquad w_0=\begin{pmatrix}&1&\\&&1\\1&&\end{pmatrix}$$

Conjugating the matrix with the r and r_1 variable to the left, after changing variables in $U(GL_3)$, we may assume that $\delta_1=0$. Thus we obtain

$$\sum_{\delta_2\in F}\theta_6^{U(GL_3),\psi}\left[w\begin{pmatrix}1&&\delta_2&&&\\&1&&&&\\&&1&&&\\&&&1&&-\delta_2\\&&&&1&\\&&&&&1\end{pmatrix}m\right]$$

Conjugating m to the left, changing variables in $U(GL_3)$ and using equation (43) implies that the above sum is in fact left invariant under $m\in \widetilde{Sp}_6(\mathbf{A})$.

From this we conclude that $L(vh)=L(v)$ for all $v\in V(2)$. Since $V(2)$ is a unipotent radical of a maximal parabolic subgroup of G_2, it follows that the integral I'' is zero by the cuspidality of $\widetilde{\pi}$.

Next we consider I'. As in the proof of Proposition 16, it follows that this integral is equal to

$$\int\limits_{G_2(F)\backslash G_2(\mathbf{A})} \widetilde{\varphi}(h)\theta^{U(B_3)}((h,g))dh$$

where $U(B_3) = U_{\alpha_1,\alpha_2,\alpha_3}$. To prove that this integral is zero for all choice of data, let $E(h,s)$ denote the Eisenstein series associated with the induced representation $Ind_{L(\mathbf{A})}^{G_2(\mathbf{A})}\delta_L^s$. Here L is the maximal parabolic subgroup of G_2 which preserves a line. Consider the integral

$$\int\limits_{G_2(F)\backslash G_2(\mathbf{A})} \widetilde{\varphi}(h)\theta^{U(B_3)}((h,1))E(h,s)dh$$

As in (51), unfolding the Eisenstein series, using Proposition 1, we show that this integral is zero for all $Re(s)$ large. Thus its residue at $s=1$ is zero, from which it follows that $I'=0$. Thus the lift is cuspidal. □

Next we shall give a criterion for the lift to be nonzero. To do that, let

$$V_1 =< x_{(1000)}(r)x_{(0010)}(r); x_{(1100)}(r)x_{(0110)}(r); x_{(1110)}(r)x_{(0120)}(r); x_{(1120)}(r); x_{(1220)}(r) >$$

Thus, V_1 is a unipotent radical of the maximal parabolic subgroup of G_2 which preserves a line. We construct a projection from V_1 to $\mathcal{H}_3$, the Heisenberg group with three variables, defined as follows. Write $v \in V_1$ as

$$v = x_{(1000)}(r_1)x_{(0010)}(r_1)x_{(1100)}(r_2)x_{(0110)}(r_2)x_{(1110)}(r_3)x_{(0120)}(r_3)x_{(1120)}(r_4)x_{(1220)}(r_5)$$

Then we define $l : V_1 \mapsto \mathcal{H}_3$ as $l(v) = (r_1, r_2, r_3)$. Here, we identify elements in $\mathcal{H}_3$ as triples, where the third coordinate is the center of $\mathcal{H}_3$. The group SL_2 generated by $< x_{\pm(0100)}(r) >$ normalizes the group V_1. We have

Proposition 20. *The representation $\sigma(\widetilde{\pi})$ is nonzero, if and only if the integral*

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})}\int\limits_{V_1(F)\backslash V_1(\mathbf{A})} \widetilde{\varphi}(vm)\theta_{SL_2}^{\phi,\psi}(l(v)m)dvdm \tag{66}$$

is nonzero for some choice of data. Here $\theta_{SL_2}^{\phi,\psi}$ is a vector in the space of $\Theta_{SL_2}^{\psi}$, the theta representation of $\mathcal{H}_3(\mathbf{A})\cdot\widetilde{SL}_2(\mathbf{A})$.

Proof. Keeping the notations in the proof of Proposition 19, the lift is nonzero for some choice of data, if and only if the integral

$$W_f(g) = \int\limits_{G_2(F)\backslash G_2(\mathbf{A})}\int\limits_{F\backslash\mathbf{A}} \widetilde{\varphi}(h)\theta((h,x(l_1)g))\psi(l_1)dl_1dh$$

is nonzero for some choice of data. Here $x(l_1)$ was defined in the beginning of the proof of Proposition 19. Arguing as in the proof of Proposition 19, we obtain

$$W_f(g) = \int\limits_{S(2)(F)\backslash G_2(\mathbf{A})}\int\limits_{(F\backslash\mathbf{A})^2} \widetilde{\varphi}(h)\theta^U(x_{0121}(l_1)x_{0122}(l_2)(h,g))\psi(l_1+l_2)dl_idh$$

Here $S(2) = V_1 \cdot SL_2$ where the SL_2 is generated by $< x_{\pm(0100)}(r) >$. Factoring the integration over $S(2)$, and plugging $g = e$, then $W_f(e)$ is equal to

$$\int\limits_{S(2)(\mathbf{A})\backslash G_2(\mathbf{A})} \int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \int\limits_{V_1(F)\backslash V_1(\mathbf{A})} \int\limits_{(F\backslash \mathbf{A})^2} \widetilde{\varphi}(v_1 m h) \times$$
$$\theta^U(x_{0121}(l_1)x_{0122}(l_2)v_1 m(h,e))\psi(l_1 + l_2)dl_i dv_1 dm dh$$

Arguing in a similar way as in [Ga-S], the above integral is zero for all choice of data if and only if the integral

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \int\limits_{V_1(F)\backslash V_1(\mathbf{A})} \int\limits_{(F\backslash \mathbf{A})^2} \widetilde{\varphi}(v_1 m)\theta^U(x_{0121}(l_1)x_{0122}(l_2)v_1 m)\psi(l_1 + l_2)dl_i dv_1 dm$$

is zero for all choice of data. From the description of V_1 in terms of roots in F_4, it follows after a change of variables in U, that the above integral is equal to

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \int\limits_{V_1(F)\backslash V_1(\mathbf{A})} \int\limits_{(F\backslash \mathbf{A})^2} \widetilde{\varphi}(v_1 m)\theta^U(x_{0121}(l_1)x_{0122}(l_2)y(m_1,m_2,m_3)m)\psi(l_1 + l_2)dl_i dv_1 dm$$

where $y(m_1, m_2, m_3) = x_{0010}(m_1)x_{0110}(m_2)x_{0120}(m_3)$. From Proposition 1, this integral is zero for all choice of data if and only if the integral

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \int\limits_{(F\backslash \mathbf{A})^7} \widetilde{\varphi}(k(m_1,m_2,m_3,r_1,r_2)m)\theta_6(z(l_1,l_2)y(m_1,m_2,m_3)m)\psi(l_1 + l_2)dl_i dm_j dm$$

is zero for all choice of data. Here $z(l_1, l_2) = I_6 + l_1(e_{1,5} + e_{2,6}) + l_2 e_{1,6}$, and $y(m_1, m_2, m_3) = I_6 + m_1(e_{2,3} - e_{4,5}) + m_2(e_{2,4} + e_{3,5}) + m_3 e_{2,5}$, both matrices in Sp_6. Also, $k(m_1, m_2, m_3, r_1, r_2)$ is equal to

$$x_{1000}(m_1)x_{0010}(m_2)x_{1100}(m_2)x_{0110}(m_2)x_{1110}(m_3)x_{0120}(m_3)x_{1120}(r_1)x_{1220}(r_2)$$

Next we use Proposition 8 to obtain that the above integral is equal to

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \int\limits_{(F\backslash \mathbf{A})^7} \widetilde{\varphi}(k(m_1,m_2,m_3,r_1,r_2)m)\theta^{\phi,\psi}_{Sp_4}(h(l_2)y(m_1,m_2,m_3)m)\psi(l_1)dl_1 dm_j dm$$

Here $\theta^{\phi,\psi}_{Sp_4}$ is a vector in the space of $\Theta^{\psi}_{Sp_4}$ which is the theta representation defined on the group $\mathcal{H}_5(\mathbf{A}) \cdot \widetilde{Sp}_4(\mathbf{A})$. The element $h(l_2)$ is an element in $\mathcal{H}_5(\mathbf{A})$ which is equal to $h(l_2) = (0, 0, 0, l_2, 0)$. Here we view elements of $\mathcal{H}_5(\mathbf{A})$ as defined in [I1]. Applying the formulas of the Weil representation, see [I1], we obtain integral (66) as inner integration. Arguing again in a similar way as in [Ga-S], the Proposition follows. □

3.5. **The Commuting Pair** (SL_2, Sp_6)**.** In this subsection we study the lifting from automorphic representations of $\widetilde{SL}_2(\mathbf{A})$ to automorphic representations of $\widetilde{Sp}_6(\mathbf{A})$, and also the lifting in the other direction.

3.5.1. **From $\widetilde{SL}_2(\mathbf{A})$ to $\widetilde{Sp}_6(\mathbf{A})$.** To study this lift we consider the following embedding of the groups SL_2 and Sp_6 in F_4. First we embed the group SL_2 as the group $< x_{\pm 0100}(r) >$. The embedding of Sp_6 is as the group generated by $< x_{\pm 0120}(r); x_{\pm 0001}(r); x_{\pm 1110}(r) >$. These two embedding do not split under the double cover. The group Sp_6 has three maximal parabolic subgroups, and we denote their unipotent radical by V_i for $1 \leq i \leq 3$. The roots inside these three unipotent groups are $\{(1110; (1111); (1231); (1232); (2342)\}$ in V_1, $\{(0001); (1111); (0121); (1231); (0122); (1232); (2342)\}$ in V_2, and $\{(0120); (0121); (0122); (1231); (1232); (2342)\}$ in V_3.

Let $\widetilde{\pi}$ denote an irreducible cuspidal representation of $\widetilde{SL}_2(\mathbf{A})$. The lift we consider is given by

$$f(g) = \int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \widetilde{\varphi}(h)\theta(((h,g))dh$$

We denote by $\sigma(\widetilde{\pi})$ the automorphic representation of $\widetilde{Sp}_6(\mathbf{A})$ generated by the above functions. The result we prove is

Proposition 21. *Let $\widetilde{\pi}$ denote an irreducible cuspidal representation of $\widetilde{SL}_2(\mathbf{A})$ which lift to a cuspidal representation of $GL_2(\mathbf{A})$. Then the representation $\sigma(\widetilde{\pi})$ is nonzero. Assume also that integral* (70) *is zero for all choice of data. Then, the constant terms of this representation along the unipotent groups V_1 and V_3 are zero.*

Proof. We start with the computation of the constant terms along the groups V_1 and V_3. Thus, for $i = 1, 3$ we need to prove that the integral

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \int\limits_{V_i(F)\backslash V_i(\mathbf{A})} \widetilde{\varphi}(h)\theta((h,v))dvdh \tag{67}$$

is zero for all choice of data. Let $Z = \{x_{2342}\}$. Then $Z \subset V_i$ and using Proposition 3 integral (67) is equal to

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \widetilde{\varphi}(h)\theta^U(((h,v))dvdh+ \tag{68}$$

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \widetilde{\varphi}(h) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\gamma\in Q(F)\backslash Sp_6(F)} \sum_{\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)\gamma(v,g))dvdh$$

The first summand is zero. Indeed, it follows from Proposition 1 that we obtain the integral

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \widetilde{\varphi}(h)\theta_6(h)dh \tag{69}$$

as inner integration. Here, the embedding of SL_2 inside Sp_6 is given by $h \to \mathrm{diag}(1, 1, h, 1, 1)$. Also, θ_6 is a vector in the space of the representation Θ_6 which was defined right before

Proposition 2. Let Z denote the subgroup of Sp_6 which was defined right before Proposition 8. Expanding the above integral along $Z(F)\backslash Z(\mathbf{A})$, we consider first the contribution from the nontrivial characters. To that we use Proposition 8 to obtain the integral

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \widetilde{\varphi}(h)\theta_{Sp_4}^{\phi',\psi_\beta}(h)dh$$

as inner integration. Applying the Theta representation properties, see [I1] and [G-R-S6], we obtain the integral

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \widetilde{\varphi}(h)\theta_{SL_2}^{\phi',\psi}(h)dh$$

It follows from the assumption on $\widetilde{\pi}$ that this integral is zero for all choice of data. Thus, integral (69) is equal to

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \widetilde{\varphi}(h)\theta_6^{Z}(h)dh$$

The quotient $U(GL_1 \times Sp_4)/Z$ is abelian. Here $U(GL_1 \times Sp_4)$ was defined right before Proposition 6. Expanding along this quotient, and using the fact that Θ_6 is a minimal representation, the above integral is equal to

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \widetilde{\varphi}(h)\theta_6^{U(GL_1\times Sp_4)}(h)dh$$

We proceed with these Fourier expansions, and using the minimality of Θ_6, we deduce that the above integral is equal to

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \widetilde{\varphi}(h)\theta_6^{V}(h)dh \tag{70}$$

Here V is the unipotent radical of the parabolic subgroup of Sp_6 whose Levi part is $GL_1^2 \times SL_2$. This integral is zero by assumption. Hence the first summand in (68) is zero for all choice of data.

Next we compute the second summand in (68). Let P denote the maximal parabolic subgroup of Sp_6 whose Levi part is $GL_1 \times Sp_4$. The space of double cosets $Q(F)\backslash Sp_6(F)/P(F)$ contains two elements, and we take e and $w[234]$ as representatives. Denote by I_1 the contribution from e and by I_2 the contribution from $w[234]$. Then

$$I_1 = \int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \widetilde{\varphi}(h) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\gamma\in S(3)(F)\backslash Sp_4(F)} \sum_{\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)\gamma(v,g))dvdh$$

Here $S(3)$ is the maximal parabolic subgroup of Sp_4 which contains the group $< x_{\pm 0010} >$. The space of double cosets $S(3)(F)\backslash Sp_4(F)/S(2)(F)$ contains two elements with representatives e and $w[23]$. Here $S(2)$ is the maximal parabolic subgroup of Sp_4 whose Levi part

contains the group $< x_{\pm 0100} >$. The contribution to I_1 from e is given by

$$\int\limits_{B(F)\backslash SL_2(\mathbf{A})} \widetilde{\varphi}(h) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)(v,g))dvdh$$

Here B is the Borel subgroup of SL_2. Using Proposition 4 this integral is zero by cuspidality of $\widetilde{\pi}$. This follows from the fact that the function $g\mapsto \theta^{U,\psi}(h_2(\epsilon)(v,g))$ is left invariant under $\{x_{0100}(r)\}$ with $r\in\mathbf{A}$. The contribution to I_1 from $w[23]$ is given by

$$\int\limits_{B(F)\backslash SL_2(\mathbf{A})} \widetilde{\varphi}(h) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\delta_i\in F}\sum_{\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[23]x_{0010}(\delta_1)x_{0120}(\delta_2)(v,g))dvdh$$

If δ_1 is zero, arguing as in the previous integral, using Proposition 4, we get zero contribution by the cuspidality of $\widetilde{\pi}$. Assume $\delta_1\neq 0$. If $i=1$, then V_1 contains the root (1110). Conjugating $x_{1110}(r)$ from right to left, using commutation relations and Proposition 4, we obtain $\int\limits_{F\backslash\mathbf{A}} \psi(\epsilon^{-1}\delta_1 r)dr$ as inner integration. Since δ_1 and ϵ are nonzero this integral is zero. When $i=3$, the group V_3, contains $\{x_{0120}(r)\}$. Collapse the summation over δ_2 with the corresponding integration, we then get that the function $h\rightarrow \int\limits_{\mathbf{A}} \theta^{U,\psi}(h_2(\epsilon)w[23]x_{0010}(\delta_1)x_{0120}(r)(1,h))dr$ is left invariant under $\{x_{0100}(m)\}$ for all $m\in\mathbf{A}$. Thus we get zero by cuspidality. Thus $I_1=0$.

We are left with I_2 which is equal to

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \widetilde{\varphi}(h) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\gamma\in S(3)(F)\backslash Sp_4(F)}$$

$$\sum_{\delta_i\in F,\ \epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[234]x_{0001}(\delta_1)x_{0011}(\delta_2)x_{0122}(\delta_3)\gamma(v,g))dvdh$$

As above, we take e and $w[23]$ as representatives for $S(3)(F)\backslash Sp_4(F)/S(2)(F)$, and so I_2 is a sum of two integrals which we denote by I_{21} and I_{22}. The integral I_{21} is equal to

$$\int\limits_{B(F)\backslash SL_2(\mathbf{A})} \widetilde{\varphi}(h) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\delta_i\in F}\sum_{\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[234]x_{0001}(\delta_1)x_{0011}(\delta_2)x_{0122}(\delta_3)(v,g))dvdh$$

If δ_2 equal zero, then we get zero by cuspidality. This follows from the fact that the function $g\mapsto \theta^{U,\psi}(h_2(\epsilon)w[234]x_{0001}(\delta_1)x_{0122}(\delta_3)(v,g))$ is left invariant by $\{x_{0100}(r)\}$ with $r\in\mathbf{A}$. Assume $\delta_2\neq 0$. The group V_1 contains the root (1111). Conjugating by $x_{1111}(r)$, we obtain $\int\limits_{F\backslash\mathbf{A}} \psi(\epsilon^{-1}\delta_2 r)dr$ as inner integration, and hence we get zero. The group V_3, contains the root (0122). As in the case of I_1, we collapse summation and integration , and then get zero by cuspidality. We are left with I_{22} which is equal to

$$\int\limits_{B(F)\backslash SL_2(\mathbf{A})} \widetilde{\varphi}(h) \int\limits_{Z(\mathbf{A})V_i(F)\backslash V_i(\mathbf{A})} \sum_{\delta_i\in F}\sum_{\epsilon\in F^*} \theta^{U,\psi}(h_2(\epsilon)w[23423]y(\delta_1,\delta_2,\delta_3,\delta_4,\delta_5)(v,g))dvdh$$

where

$$y(\delta_1,\delta_2,\delta_3,\delta_4,\delta_5)=x_{0010}(\delta_1)x_{0011}(\delta_2)x_{0120}(\delta_3)x_{0121}(\delta_4)x_{0122}(\delta_5)$$

Both unipotent subgroups contains the two roots (1231) and (1232). We have

$$x_{0011}(\delta_2)x_{1231}(r)=x_{1242}(r\delta_2)x_{1231}(r)x_{0011}(\delta_2)$$

Since $w[23423]$ conjugates the root (1242) to (1000), it follows that if $\delta_2\neq 0$, then the contribution to the above integral is zero. Indeed, using these commutation relations, and a change of variables in U, we obtain the integral $\int\psi(\delta_2\epsilon r)dr$ as inner integration. Similarly, using the root (1232) we deduce that the contribution from $\delta_1\neq 0$ is zero. When $\delta_1=\delta_2=0$ we once again use the left invariance of $\theta^{U,\psi}$ by $x_{0100}(r)$ with $r\in\mathbf{A}$, to get zero by cuspidality. This completes the proof that the constant terms along the unipotent groups V_1 and V_3 are zero.

Next we consider the question of the non vanishing of the lift. We will prove that there is a choice of data such that the integral

$$\int\limits_{(F\backslash\mathbf{A})^6} f(x_{0120}(r_1)x_{0121}(r_2)x_{0122}(r_3)x_{1231}(r_4)x_{1232}(r_5)x_{2342}(r_6))\psi(\beta r_1+\gamma r_3+r_6)dr_i \tag{71}$$

is not zero for some $\beta,\gamma\in F^*$. Assume not. Then, for all β and γ and all choice of data, this integral is zero. Plugging this into the definition of the lift, we deduce that for all choice of data, the integral

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})}\widetilde{\varphi}(h)\int\limits_{V(F)\backslash V(\mathbf{A})}\theta((h,v))\psi_{V,\beta,\gamma}(v)dvdh$$

is zero. Here we wrote V and $\psi_{V,\beta,\gamma}$ for the group generated by the 6 roots in (71) and for the character of this group. The group V is abelian. Let $U_1=\{x_{1231}(r),x_{1232}(r)\}$ and $U_2=\{x_{0120}(r),x_{0121}(r),x_{0122}(r)\}$. Using Proposition 5 we deduce that for all choice of data, the integral

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})}\widetilde{\varphi}(h)\int\limits_{U_1(F)\backslash U_1(\mathbf{A})}\int\limits_{U_2(F)\backslash U_2(\mathbf{A})}\theta^{\phi,\psi}_{Sp_{14}}(\iota(u_1)\varpi_3(u_2h))\psi_{U_2,\beta,\gamma}(u_2)du_1du_2dh$$

is zero. We describe the embedding of the various groups inside the Heisenberg group $\mathcal{H}_{15}$ and in Sp_{14}. We use the parametrization as described in integral (71). First, inside the Heisenberg group we have

$$x_{1231}(r_4)x_{1232}(r_5)=(0,\ldots,0,r_4,r_5,0,0,0)$$

where the last coordinate is the center of the Heisenberg group. Here we identify the group $\mathcal{H}_{15}$ with a 15 tuple. See [I1]. Next

$$x_{0120}(r_1)=I_{14}+r_1e'_{1,5}+r_1e'_{2,6}+r_1e_{4,11},\quad x_{0122}(r_3)=I_{14}+r_3e'_{1,9}+r_3e'_{2,10}+r_3e_{3,12}$$

Here $e_{i,j}$ denotes the matrix of size 14 which has one at the (i,j) entry and zero elsewhere, and $e'_{i,j}=e_{i,j}-e_{15-j,15-i}$. The above two matrices are in Sp_{14}. Also, we have

$$x_{0121}(r_2)=I_{14}+r_2e'_{1,7}+r_2e'_{2,8}+r_2e'_{3,11}$$

Finally, the group SL_2 is embedded in Sp_{14} as $h\to \text{diag}(h,I_2,h,h,h^*,I_2,h^*)$.

In the above integral we unfold the theta function. Using the action of the Weil representation under the Heisenberg group, see [G-R-S6], we deduce that $\xi_3=\xi_4=0$. Thus the integral

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})}\widetilde{\varphi}(h)\int\limits_{(F\backslash \mathbf{A})^3}$$

$$\sum_{\xi_i\in F}\omega_\psi(x_{0120}(r_1)x_{0121}(r_2)x_{0122}(r_3)h)\phi(\xi_1,\xi_2,0,0,\xi_5,\xi_6,\xi_7)\psi(\beta r_1+\gamma r_3)dr_idh$$

is zero for all choice of data. From the embedding of the group SL_2 in Sp_{14}, and from the action of the Weil representation, we obtain that the group $SL_2(F)$ acts on the first two coordinates ξ_1 and ξ_2 with two orbits. The trivial orbit contributes zero. Indeed, from the embedding of the unipotent group $\{x_{0120}(r_1)\}$ inside Sp_{14}, we obtain the integral $\int\psi(\beta r_1)dr_1$ as inner integration. Thus the above integral is equal to

$$\int\limits_{N(F)\backslash SL_2(\mathbf{A})}\widetilde{\varphi}(h)\int\limits_{(F\backslash \mathbf{A})^3}$$

$$\sum_{\xi_i\in F}\omega_\psi(x_{0120}(r_1)x_{0121}(r_2)x_{0122}(r_3)h)\phi(0,1,0,0,\xi_5,\xi_6,\xi_7)\psi(\beta r_1+\gamma r_3)dr_idh$$

which is zero for all choice of data. Here N is the unipotent radical of the Borel subgroup of SL_2. Applying the integration over r_2 and then over r_3, and arguing as above, we deduce that $\xi_7=0$ and $\xi_5=\gamma$. Collapsing the summation over ξ_6 with the integration over r_1 we obtain that the integral

$$\int\limits_{N(F)\backslash SL_2(\mathbf{A})}\widetilde{\varphi}(h)\int\limits_{\mathbf{A}}\omega_\psi(h)\phi(0,1,0,0,\gamma,r_1,0)\psi(\beta r_1)dr_1dh$$

is zero for all choice of data. Finally, factoring the integration over N we obtain that the integral

$$\int\limits_{N(\mathbf{A})\backslash SL_2(\mathbf{A})}W_{\widetilde{\varphi}}^{\psi,\beta,\gamma}(h)\int\limits_{\mathbf{A}}\omega_\psi(h)\phi(0,1,0,0,\gamma,r_1,0)\psi(\beta r_1)dr_1dh$$

is zero for all choice of data. Here

$$W_{\widetilde{\varphi}}^{\psi,\beta,\gamma}(h)=\int\limits_{F\backslash\mathbf{A}}\widetilde{\varphi}\left(\begin{pmatrix}1&y\\&1\end{pmatrix}h\right)\psi(-\beta\gamma y)dy$$

Using a similar argument as in [Ga-S] we deduce that $W_{\widetilde{\varphi}}^{\psi,\beta,\gamma}(h)$ is zero for all β and γ. This is clearly a contradiction.

$\square$

3.5.2. **From $\widetilde{Sp}_6$ to $\widetilde{SL}_2$.** In this case we choose the following embedding of the two groups. First, the Sp_6 is generated by $< x_{\pm 0100}(r); x_{\pm 0010}(r); x_{\pm 0001}(r) >$, and the SL_2 is generated by $< x_{\pm 2342}(r) >$. This embedding is conjugated by the Weyl element $w[3124321]$ to the embedding of the two groups as was described in the previous subsection.

Let $\widetilde{\pi}$ denote an irreducible cuspidal representation of $\widetilde{Sp}_6(\mathbf{A})$. The lift we consider is

$$f(g) = \int\limits_{Sp_6(F)\backslash Sp_6(\mathbf{A})} \widetilde{\varphi}(h)\theta((h,g))dh$$

We prove the following

Proposition 22. *Let $\sigma(\widetilde{\pi})$ denote the automorphic representation of $\widetilde{SL}_2$ generated by the above functions. Then $\sigma(\widetilde{\pi})$ is a cuspidal representation. It is nonzero if and only if the integral*

$$\int\limits_{Sp_6(F)\backslash Sp_6(\mathbf{A})} \widetilde{\varphi}(h)\theta_{Sp_{14}}^{\phi,\beta}(\varpi_3(h))dh$$

is nonzero for some choice of data. Here $\beta \in F^*$.

Proof. The representation $\sigma(\widetilde{\pi})$ is nonzero if and only if the integral

$$\int\limits_{Sp_6(F)\backslash Sp_6(\mathbf{A})} \int\limits_{F\backslash \mathbf{A}} \widetilde{\varphi}(h)\theta((h, x_{2342}(r)))\psi(\beta r)drdh$$

is nonzero for some choice of data. Thus the claim about the nonvanishing follows from Proposition 5.

As for the cuspidality, we use Proposition 3 to write the constant term of the SL_2 as

$$\int\limits_{Sp_6(F)\backslash Sp_6(\mathbf{A})} \widetilde{\varphi}(h)\theta^{U}((h,1))dh + \int\limits_{Q^0(F)(F)\backslash Sp_6(\mathbf{A})} \widetilde{\varphi}(h)\theta^{U,\psi}((h,1))dh$$

where Q^0 is the subgroup of Q, which is the semidirect product of SL_3 and the unipotent radical of Q, the group $U(Q)$. (See Proposition 6). The first summand is zero because of Proposition 1. As for the second summand, it follows from Proposition 4 that the function $h \mapsto \theta^{U,\psi}((h,1))$ is left invariant by the group $U(Q)(\mathbf{A})$. Thus, we obtain zero by the cuspidality of $\widetilde{\pi}$.

$\square$

3.6. **The Liftings as a Functorial Lifting.** The next step, and an important one, is to determine which of the above constructions defines a functorial liftings. It is also an interesting problem to see if each of these pairs satisfy the unramified Howe duality property.

More precisely, let (H, G) be one of the above commuting pair. Thus, if π is a cuspidal irreducible representation of $G(\mathbf{A})$, or its double cover, and if σ is a cuspidal irreducible representation of $H(\mathbf{A})$, or its double cover, we are interested in the cases when the global integral

$$\int\limits_{H(F)\backslash H(\mathbf{A})} \int\limits_{H(F)\backslash H(\mathbf{A})} \varphi_\sigma(h)\varphi_\pi(g)\theta((h,g))dhdg \tag{72}$$

is not zero for some choice of data. Here φ_σ is a vector in the space of σ, φ_π is a vector in the space of π, and θ is a vector in the space of Θ. Following [G-R-S4] pages 606-608, then the nonvanishing of the integral (72) implies that at any local place there is a nonzero such a trilinear form. In other words, let ν be a place where all representations are unramified. Let $\sigma_\nu = Ind_{B(H)}^H \chi$ denote an unramified representation of H or its double cover, at the place ν. When there is no confusion, we shall omit ν from the notations. Similarly, let $\pi_\nu = Ind_{B(G)}^G \mu$ denote an unramified representation of G or its double cover at the place ν. Then we assume that the space

$$Hom_{G\times H}(Ind_{B(G)}^G \mu \times Ind_{B(H)}^H \chi, \theta)$$

is not zero. Here θ is the local unramified constituent of Θ at the place ν. The unramified Howe duality property states that given χ and μ as above, then each one of these characters determine uniquely the other.

Conjecture: *All the five commuting pairs, which were described in the beginning of this Section, satisfy the local unramified Howe duality property.*

In each of the five cases we studied we will now give a conjectural description of the lift.

1) (SL_3, SL_3). Here the construction is from the space of cuspidal representation defined on $\widetilde{GL}_3(\mathbf{A})$ to the space of automorphic representations of $SL_3(\mathbf{A})$. The conjectural functorial lift is the well known Shimura lift. Some information at the role of the orthogonal period which we obtained can be found in [J2].

2) $(SL_2 \times SL_2, Sp_4)$. Here the conjectural lift is the endoscopic lift. In more details, the corresponding L groups of $\widetilde{SL}_2(\mathbf{A})$ and $\widetilde{Sp}_4(\mathbf{A})$ are $SL_2(\mathbf{C})$ and $Sp_4(\mathbf{C})$. Hence, the conjecture lift in this case is corresponding to the homomorphism from $SL_2(\mathbf{C}) \times SL_2(\mathbf{C})$ into $Sp_4(\mathbf{C})$. This lift is a special case of the more general construction as studied in [G-R-S7].

3) (SL_2, SL_4). The conjectural lift in this case is a special case of the conjecture stated in [S]. We state it for our case. Let π denote an irreducible cuspidal representation of $GL_2(\mathbf{A})$. Suppose that π is a functorial lift from $\widetilde{GL}_2(\mathbf{A})$ which is given by the Shimura lift.

Then it is conjectured in [S] that π has a nontrivial lift to $\widetilde{GL}_4(\mathbf{A})$ and that the image is a certain residue representation. If π is not in the image of the Shimura correspondence, then the conjecture states that π has a nontrivial functorial lift to a cuspidal representation of $\widetilde{GL}_4(\mathbf{A})$. Thus, we conjecture that this commuting pair will yield this lift. We remark that the conjecture stated in [S] is for all cuspidal representations of $GL_n(\mathbf{A})$.

4) (SO_3, G_2). This is an extension of the Classical Theta lift in the symplectic groups. In details, let π denote a cuspidal irreducible representation of $SO_3(\mathbf{A})$. As follows from [R], if the lift to $\widetilde{SL}_2(\mathbf{A})$ is zero, then the lift to $\widetilde{Sp}_4(\mathbf{A})$ is a generic cuspidal representation. Here the lift is obtained using the minimal representation of $\widetilde{Sp}_{2n}(\mathbf{A})$ where $n = 3, 6$. The conjecture in this case is that the same phenomena occurs with the exceptional group G_2 replacing the group Sp_4. In other words, if the lift of π to $\widetilde{SL}_2(\mathbf{A})$ is zero, then the lift to $\widetilde{G}_2(\mathbf{A})$ is a generic cuspidal representation.

5) (SL_2, Sp_6). In this case we showed that the image is not cuspidal. We conjecture that we obtain a residual representation of $\widetilde{Sp}_6(\mathbf{A})$ which we now describe. Let π denote an irreducible cuspidal representation of $\widetilde{SL}_2(\mathbf{A})$. Suppose that π has a functorial lift to a cuspidal representation τ of $GL_2(\mathbf{A})$. Then the partial tensor product L function $L_\psi(\pi\times\tau, s')$ has a simple pole at $s' = 1$. Form the Eisenstein series $E_{\tau,\pi}(g, s)$ defined on $\widetilde{Sp}_6(\mathbf{A})$, which is associated with the induced representation $Ind_{Q'(\mathbf{A})}^{\widetilde{Sp}_6(\mathbf{A})}(\tau \times \pi)\delta_Q^s$. Here Q' is the subgroup of $\widetilde{Sp}_6$ defined as follows. Let Q denote the maximal parabolic subgroup of Sp_6 whose Levi part is $GL_2 \times SL_2$. Let $U(Q)$ denote its unipotent radical. Then $Q' = (GL_2 \times \widetilde{SL}_2)U(Q)$. Its not hard to check that this Eisenstein series has a simple pole at s_0 corresponding to the point $s' = 1$. If we denote the residue representation by $\mathcal{E}_{\tau,\pi}$, then the conjecture is that this is the representation obtained in this case.

4. **Global Split Descent Constructions**

In this Section we consider some global descent constructions. We briefly recall the setup for this construction in the context of the group F_4 (for classical groups see [G-R-S7]). Let $\mathcal{O}$ denote a unipotent orbit of F_4. It follows from [C], that the stabilizer of each such orbit inside a suitable Levi subgroup, is a reductive group. As explained in Section 2, to each such orbit we can associate a set of Fourier coefficients. Thus, to each such orbit, we attach a unipotent group U_Δ, and a set of characters ψ_{U,u_Δ}. Let H denote the connected component of the reductive part of the stabilizer of the character ψ_{U,u_Δ}. In this paper we will only consider those characters ψ_{U,u_Δ} such that the group H is split. In some cases, one can also consider characters such that the stabilizer is an anisotropic group. However, for the analysis of when the lift is cuspidal and the study of the Fourier coefficients of the lift, the split case

is the hardest case and the more interesting one. Hence, we refer to these constructions as split descent constructions.

Let $\mathcal{E}$ denote an automorphic representation of the group F_4. In principle there is no reason not to consider also automorphic representation defined on metaplectic covering of the group F_4. To avoid issues related to cocyles, we shall restrict to representations of F_4 only. There are two cases to consider. The first, is when the Dynkin diagram attached to the unipotent orbit $\mathcal{O}$ is a diagram whose all nodes are labeled with zeros and twos. In the notations of subsection 2.2 we have in this case $U_\Delta = U_\Delta(2)$. In this case we consider the space of functions

$$f(h) = \int\limits_{U_\Delta(F)\backslash U_\Delta(\mathbf{A})} E(uh)\psi_{U,u_\Delta}(u)du \tag{73}$$

Here, E is a vector in the space of $\mathcal{E}$. Thus, $f(h)$ defines an automorphic function on the group $H(\mathbf{A})$. We denote by σ the representation of $H(\mathbf{A})$ generated by all functions $f(h)$. We refer to the representation σ as the descent representation of $\mathcal{E}$. If the representation $\mathcal{E}$ depends on an automorphic representation τ of another group, we sometimes refer to σ as the descent representation from τ.

The second case is when the diagram attached to the unipotent orbit contains also ones. In this case $U_\Delta = U_\Delta(1) \neq U_\Delta(2)$. In other words, the set $U'_\Delta(1)$ is not empty. Therefore, there is a projection from the group U_Δ onto a suitable Heisenberg group. In particular the stabilizer H has an embedding into a suitable symplectic group. In [G-R-S3] there is a detailed discussion of this situation for unipotent orbits of the symplectic groups. However, these ideas hold for any algebraic group. In this case we consider the integral

$$f(h) = \int\limits_{U_\Delta(F)\backslash U_\Delta(\mathbf{A})} \widetilde{\theta}_{Sp}^{\psi,\phi}(l(u)h)E(uh)\widetilde{\psi}_{U,u_\Delta}(u)du \tag{74}$$

Here l denotes the projection from U_Δ onto the Heisenberg group. The function $\widetilde{\theta}_{Sp}^{\psi,\phi}$ is a vector in $\widetilde{\Theta}_{Sp}^{\psi}$, the minimal representation of the double cover of the suitable symplectic group. The character $\widetilde{\psi}_{U,u_\Delta}$ is defined such that when combined with the character of the theta function it produces the character ψ_{U,u_Δ}. For more details see [G-R-S3] page 4 formula (1.3). The function $f(h)$ defined in (74) is left invariant under the rational points of H. However, depending on the embedding of H inside the symplectic group, it may be a genuine function on $\widetilde{H}(\mathbf{A})$, the double cover of $H(\mathbf{A})$.

By unfolding the theta function in integral (74) we may associate with this integral two more integrals which are related to the unipotent orbit $\mathcal{O}$. The relation, as explained in details in [G-R-S3] Lemma 1.1, is that one integral is zero for all choice of data if and only if

the other is zero for all choice of data. We briefly explain the relation. Denote $U''_\Delta = U_\Delta(2)$. The second integral which is related to (74) is

$$\int\limits_{U''_\Delta(F)\backslash U''_\Delta(\mathbf{A})} E(u''h)\psi_{U,u_\Delta}(u'')du'' \tag{75}$$

where the character ψ_{U,u_Δ} was defined in subection 2.2. The third related integral is defined as follows. Consider the set of roots $U'_\Delta(1)$. Then there is a choice, in fact more than one choice, to extend the group U''_Δ to a unipotent group U'_Δ such that $U''_\Delta \subset U'_\Delta \subset U_\Delta$ and which satisfies the following. The extension of U'_Δ is obtained by adding half of the roots in $U'_\Delta(1)$ to U''_Δ in such a way that the character ψ_{U,u_Δ} is extended trivially to U'_Δ. The integral we then consider is

$$\int\limits_{U'_\Delta(F)\backslash U'_\Delta(\mathbf{A})} E(u'h)\psi_{U,u_\Delta}(u')du' \tag{76}$$

These two last integrals were denoted in [G-R-S3] by (1.1) and (1.2). Lemma 1.1 in that reference states that if one of these three integrals is zero for all choice of data, then the other two also vanish for all choice of data. The proof is formal and applies to all algebraic groups.

We illustrate this by an example. Consider the unipotent orbit A_1. Its diagram is

$$\overset{1}{0} - - - - 0 ==>== 0 - - - - 0$$

In this case $U_\Delta = U_{\alpha_2,\alpha_3,\alpha_4}$ is the unipotent radical of the maximal parabolic subgroup of F_4 whose Levi part is GSp_6. Also, U_Δ is isomorphic to $\mathcal{H}_{15}$, the Heisenberg group consisting of 15 variables and we denote by l this isomorphism. Hence, integral (74) is given by

$$f(h) = \int\limits_{U_\Delta(F)\backslash U_\Delta(\mathbf{A})} \widetilde{\theta}_{Sp_{14}}^{\psi,\phi}(l(u)h)E(uh)du \tag{77}$$

where $\widetilde{\theta}_{Sp_{14}}^{\psi,\phi}$ is a vector in the minimal representation of $\widetilde{Sp}_{14}(\mathbf{A})$. The connected component of the stabilizer of this unipotent orbit is the group Sp_6. In this case the automorphic function $f(h)$, and the representation σ defines a genuine automorphic function and an automorphic representation on the group $\widetilde{Sp}_6(\mathbf{A})$. Since $U_\Delta(2) = \{x_{2342}(r)\}$ (see subsection 2.1 for notations), then in this example, integral (75) is

$$\int\limits_{F\backslash \mathbf{A}} E(x_{2342}(r)h)\psi(r)dr \tag{78}$$

To describe integral (76) we need to choose half of the roots in $U'_\Delta(1)$, in such a way that we can extend the character from $\{x_{2342}(r)\}$ trivially. The choice of these roots is not unique.

For example, one can choose the following roots

$$A=\{(1122);\ (1221);\ (1222);\ (1231);\ (1232);\ (1242);\ (1342)\}$$

Thus, the group U'_Δ is the unipotent group generated by $\{x_\alpha(r)\}$ where $\alpha \in A$ together with the root (2342). The character ψ_{U,u_Δ} is defined as follows. For $u' = x_{2342}(r)u'_1$ set $\psi_{U,u_\Delta}(u') = \psi(r)$ (see subsection 2.1). Thus, ψ_{U,u_Δ} is the trivial extension of the character given in (78) from U''_Δ to U'_Δ.

As mentioned in the introduction, our goal is to look for those unipotent orbits, such that the integrals which define the descent satisfies the dimension identity (4). There are two cases. When the nodes of the diagram attached to the unipotent orbit consists of zeros and twos, then the descent is given by integral (73). In this case, since the representation π, as defined in the introduction, is trivial, the dimension identity we consider is

$$\dim\,\mathcal{E} = \dim\,U_\Delta + \dim\,\sigma \tag{79}$$

If the diagram contains also ones, then the descent is given by integral (74). In this case we also need to take into account the theta representation on the symplectic group. Thus we obtain

$$\dim\,\mathcal{E} + \dim\,\widetilde{\Theta}_{Sp}^{\psi} = \dim\,U_\Delta + \dim\,\sigma$$

We have $\dim\,\widetilde{\Theta}_{Sp}^{\psi} = \frac{1}{2}(\dim\,\mathcal{H}-1)$ where $\mathcal{H}$ is the corresponding Heisenberg group. Since this number is equal to a half of the roots in $U'_\Delta(1)$, we obtain that the dimension formula for this case is given by

$$\dim\,\mathcal{E} = \dim\,U'_\Delta + \dim\,\sigma \tag{80}$$

where the group U'_Δ was defined above.

We remark that in both cases one can show that the dimension of U_Δ in the first case, and the dimension of U'_Δ in the second case is equal to half of the dimension of the unipotent orbit in question as listed in [C-M] page 128. Thus, if we denote this unipotent orbit by $\mathcal{O}$, then equations (79) and (80) are given by

$$\dim\,\mathcal{E} = \frac{1}{2}\dim\,\mathcal{O} + \dim\,\sigma \tag{81}$$

4.1. **The dimensions for F_4 Descents.** In this subsection we consider all possible unipotent orbits such that either integral (73) or integral (74) satisfies the dimension identity (81). The list of the unipotent orbits and their stabilizers can be found in [C]. We only consider those orbits whose stabilizer contains a nontrivial reductive group. The dimension of U_Δ or U'_Δ, which is half of the dimension of the corresponding unipotent orbit, can be found in [C-M] page 128.

4.1.1. **The Unipotent Orbits C_3 and B_3.** For these orbits the stabilizer is a group of type A_1. Thus the representation σ is defined on that group, and hence $\dim \sigma = 1$. Since the dimension of the two orbits is 42, then $\frac{1}{2}\dim \mathcal{O} = 21$. Hence we look for representations $\mathcal{E}$ such that $\dim \mathcal{E} = 21 + 1 = 22$.

4.1.2. **The Unipotent Orbit $C_3(a_1)$.** The stabilizer is a group of type A_1. The dimension of this unipotent orbit is 38, and hence $\frac{1}{2}\dim \mathcal{O} = 19$. Thus $\dim \mathcal{E} = 19 + 1 = 20$.

4.1.3. **The Unipotent Orbit $\widetilde{A}_2 + A_1$.** The stabilizer is a group of type A_1, and the dimension of $\frac{1}{2}\dim \mathcal{O}$ is 18. We have $\dim \mathcal{E} = 18 + 1 = 19$.

4.1.4. **The Unipotent Orbit B_2.** The stabilizer is a group of type $A_1 \times A_1$. The dimension of $\frac{1}{2}\dim \mathcal{O}$ is 18, and hence $\dim \mathcal{E} = 18 + 2 = 20$.

4.1.5. **The Unipotent Orbit $A_2 + \widetilde{A}_1$.** The stabilizer is a group of type A_1. The dimension of $\frac{1}{2}\dim \mathcal{O}$ is 17. Hence $\dim \mathcal{E} = 18$.

4.1.6. **The Unipotent Orbit $\widetilde{A}_2$.** Here the stabilizer is the exceptional group G_2. The dimension of $\frac{1}{2}\dim \mathcal{O}$ is 15. Cuspidal representations σ on $G_2(\mathbf{A})$ can be generic, and in this case $\dim \sigma = 6$, or, if not generic, they are associated to the unipotent orbit $G_2(a_1)$. In this case $\dim \sigma = 5$. Thus, there are two cases to consider. The first is $\dim \mathcal{E} = 21$, and the second $\dim \mathcal{E} = 20$.

4.1.7. **The Unipotent Orbit A_2.** Here the stabilizer is a group of type A_2. The dimension of $\frac{1}{2}\dim \mathcal{O}$ is 15, and hence $\dim \mathcal{E} = 18$.

4.1.8. **The Unipotent Orbit $A_1 + \widetilde{A}_1$.** The stabilizer is a group of type $A_1 \times A_1$. As $\frac{1}{2}\dim \mathcal{O} = 14$ in this case, then $\dim \mathcal{E} = 16$. As it follows from [C-M] there is no unipotent orbit whose dimension is 32. Hence we do not expect that a suitable $\mathcal{E}$ will exist in this case.

4.1.9. **The Unipotent Orbit $\widetilde{A}_1$.** The stabilizer is a group of type A_3, the dimension of $\frac{1}{2}\dim \mathcal{O}$ is 11, and hence $\dim \mathcal{E} = 11 + 6 = 17$.

4.1.10. **The Unipotent Orbit A_1.** Here the stabilizer is the group Sp_6. Cuspidal representations on Sp_6 can be attached to one of the unipotent orbits, $(6), (42)$ or (2^3). Their dimensions are $9, 8$ and 6. The dimension of $\frac{1}{2}\dim \mathcal{O}$ is 8, and hence we expect $\dim \mathcal{E} = 17, 16$ or 14. As mentioned above we do not expect that a representation of dimension 16 exists for the group F_4.

4.2. **How to Compute Descents.** In this subsection we give some general remarks on how to compute a descent integral. More precisely, a typical computation of a descent construction consists of two type of computations. The first is the computation of all constant terms of the representation σ corresponding to unipotent radicals of maximal parabolic subgroups of H, and the second is a computation of a certain Fourier coefficient. The first computation is done to determine conditions when the descent is cuspidal, and the second is done to determine when the descent is nonzero. Usually, the computation of the constant term is harder since it involve many unipotent orbits. In this Section we will only consider the computation of a certain Fourier coefficient of the descent. However, we will say a few words on the computation of the constant terms at the end of the next subsection.

Let $\mathcal{O}$ be a unipotent orbit, and let $\mathcal{E}$ be an automorphic representation defined on the group $F_4(\mathbf{A})$. The group U'_Δ was defined for unipotent orbits whose diagram contains nodes labelled with the number one. It is convenient to extend the definition of the group U'_Δ to unipotent orbits whose diagrams contain nodes labelled with zeros and twos only. In this case we denote $U'_\Delta = U_\Delta$. In this way we defined the group U'_Δ for all unipotent orbits. From the discussion in the previous subsections, we are led to consider integrals of the type

$$\int\limits_{V(F)\backslash V(\mathbf{A})} \int\limits_{U'_\Delta(F)\backslash U'_\Delta(\mathbf{A})} E(uvh)\psi_{U,u_\Delta}(u)\psi_V(v)dvdu \tag{82}$$

The group V is a certain unipotent subgroup of the stabilizer of the character ψ_{U,u_Δ}. The character ψ_V is a character, possibly the trivial one, of the group $V(F)\backslash V(\mathbf{A})$.

4.2.1. **Unipotent Orbits and Torus Elements.** It is convenient to express things in more generality. Let G be an algebraic reductive group, and let $\mathcal{O}_G$ denote a unipotent orbit for G. As explained in Section 2 for the group $G = F_4$, and in [G1] for an arbitrary classical group, to this orbit we associate a unipotent subgroup $U(\mathcal{O}_G)$ of G, and a set of characters $\psi_{U(\mathcal{O}_G),u_0}$ of this group. Here u_0 is an element in the unipotent orbit $\mathcal{O}_G$ which defines the character. Given an automorphic representation $\mathcal{E}$ of G, we shall denote by $\mathcal{O}_{G,u_0}(\mathcal{E})$ the Fourier coefficient given by

$$f(h) = \int\limits_{U(\mathcal{O}_G)(F)\backslash U(\mathcal{O}_G)(\mathbf{A})} E(uh)\psi_{U(\mathcal{O}_G),u_0}(u)du \tag{83}$$

If H is a reductive group contained in the stabilizer of this unipotent orbit, then the function $f(h)$ is an automorphic function of $H(\mathbf{A})$. Let $\sigma(\mathcal{E})$ denote the automorphic representation of $H(\mathbf{A})$ generated by all the functions $f(h)$ in (83). If σ is an arbitrary automorphic representation of H, then given a unipotent orbit $\mathcal{O}_H$, then as for the group G, we shall

denote by $\mathcal{O}_{H,v_0}(\sigma)$ the Fourier coefficient

$$\int\limits_{V(\mathcal{O}_H)(F)\backslash V(\mathcal{O}_H)(\mathbf{A})} \varphi_\sigma(v)\psi_{V(\mathcal{O}_H),v_0}(v)dv$$

Here $V(\mathcal{O}_H)$ is the unipotent subgroup of H which correspond to the unipotent orbit $\mathcal{O}_H$. Similarly, $\psi_{V(\mathcal{O}_H),v_0}$ is the character attached to a representative v_0 of this orbit.

One of the goals of the descent method is to compute the integral

$$\int\limits_{V(\mathcal{O}_H)(F)\backslash V(\mathcal{O}_H)(\mathbf{A})} f(v)\psi_{V(\mathcal{O}_H),v_0}(v)dv = \tag{84}$$

$$\int\limits_{V(\mathcal{O}_H)(F)\backslash V(\mathcal{O}_H)(\mathbf{A})}\ \int\limits_{U(\mathcal{O}_G)(F)\backslash U(\mathcal{O}_G)(\mathbf{A})} E(uv)\psi_{U(\mathcal{O}_G),u_0}(u)\psi_{V(\mathcal{O}_H),v_0}(v)dvdu$$

This is a certain Fourier coefficient defined on an automorphic function E which lies in the space of a representation $\mathcal{E}$ of the group $G(\mathbf{A})$. We shall denote it by $\mathcal{O}_{G,u_0}(\mathcal{E})\circ\mathcal{O}_{H,v_0}(\sigma(\mathcal{E}))$. Thus, the goal is to express this Fourier coefficient in term of Fourier coefficients attached to unipotent orbits of G. However, it is possible that we will also obtain some constant terms in the course of this computation. Let $P = MU$ denote a parabolic subgroup of G. The constant term

$$E^U(m) = \int\limits_{U(F)\backslash U(\mathbf{A})} E(um)du$$

defines an automorphic representation of $M(\mathbf{A})$. We shall denote this representation by $\mathcal{E}^U$. If $\mathcal{O}_M$ is a unipotent orbit of M, we shall denote by $\mathcal{CT}_{G,P}[\mathcal{O}_{M,l_0}(\mathcal{E}^U)]$ the Fourier coefficient

$$\int\limits_{L(\mathcal{O}_M)(F)\backslash L(\mathcal{O}_M)(\mathbf{A})} E^U(l)\psi_{L(\mathcal{O}_M),l_0}dl$$

Here $L(\mathcal{O}_M)$ is the unipotent subgroup of M which correspond to the unipotent orbit $\mathcal{O}_M$. Thus, to express integral (84) in term of Fourier coefficients attached to unipotent orbits of G, and to Fourier coefficients associated with constant terms along certain unipotent radicals of some parabolic subgroups of G, is to determine an identity of the type

$$\mathcal{O}_{G,u_0}(\mathcal{E})\circ\mathcal{O}_{H,v_0}(\sigma(\mathcal{E})) = \sum_i(\mathcal{O}_i)_{G,u_i}(\mathcal{E}) + \sum_j \mathcal{CT}_{G,P_j}[(\mathcal{O}_j)_{M,l_j}(\mathcal{E}^U)] \tag{85}$$

In words, the goal is to express the Fourier coefficient defined by integral (84), as a sum of two type of integrals. The first term is a sum of Fourier coefficients which corresponds to unipotent orbits of the group G. Thus, we want to determine the precise unipotent orbits $\mathcal{O}_i$ appearing in the first sum. The second term on the right hand side of equation (85) is a sum of constant terms corresponding to unipotent radicals of certain parabolic subgroups of

G. In this case we need to determine which parabolic subgroup are involved, and also what are the unipotent orbits of the group M which are involved.

The identity (85) is completely formal in the sense that it does not require any information on the representation $\mathcal{E}$. From this point of view, this identity can be viewed as a formal identity defined on the level of unipotent orbits only.

To make things clear we consider an example in the group Sp_4. We refer the reader to [G-R-S6] for notations. In [G-R-S2] the descent map from GL_{2n} to $\widetilde{Sp}_{2n}$ was introduced. To do that one uses an automorphic representation of Sp_{4n} which is defined as a residue of a certain Eisenstein series. Consider the more general case of the descent when $n=1$. Thus we consider the following integral

$$f(g)=\int\limits_{(F\backslash\mathbf{A})^2}\widetilde{\theta}_{\psi^\beta,\phi}((r,x,y)g)E\left(\begin{pmatrix}1&r&x&y\\&1&&x\\&&1&-r\\&&&1\end{pmatrix}\begin{pmatrix}1&&\\&g&\\&&1\end{pmatrix}\right)dxdydrdg \tag{86}$$

Here $\beta\in F^*$ and E is a vector in some automorphic representation $\mathcal{E}$ of $Sp_4(\mathbf{A})$. Thus $f(g)$ is an automorphic representation of $\widetilde{SL}_2(\mathbf{A})$. We denote by $\sigma(\mathcal{E})$ the representation of this group generated by all functions $f(g)$. To study when it is nonzero, we compute the integral

$$\int\limits_{F\backslash\mathbf{A}}f\begin{pmatrix}1&z\\&1\end{pmatrix}\psi(\gamma z)dz$$

where $\gamma\in F^*$. As stated above, integral (74) is zero for all choice of data if and only if integral (76) is zero for all choice of data. For the group Sp_4, this was proved in [G-R-S3].

For the group Sp_4 the analogues to integral (76) is the integral

$$\int\limits_{F\backslash\mathbf{A}}\int\limits_{(F\backslash\mathbf{A})^2}E\left(\begin{pmatrix}1&&x&y\\&1&&x\\&&1&\\&&&1\end{pmatrix}\begin{pmatrix}1&&&\\&1&z&\\&&1&\\&&&1\end{pmatrix}\right)\psi(\beta y+\gamma z)dxdydz \tag{87}$$

Here $\gamma\in(F^*)^2\backslash F^*$. The x,y integration, which in the notation of integral (82) correspond to the group U'_Δ, is a Fourier coefficient corresponding to the unipotent orbit of Sp_4 associated with the partition (21^2), and the z integration is the Whittaker coefficient of $\sigma(\mathcal{E})$, defined on $\widetilde{SL}_2(\mathbf{A})$, and hence is associated with the unipotent orbit (2). Thus, in the above notations, the left hand side of (85) is $(21^2)_{Sp_4,\beta}(\mathcal{E})\circ(2)_{\widetilde{Sl}_2,\gamma}(\sigma(\mathcal{E}))$. As explained in [G1], the above integral corresponds to the unipotent orbit of Sp_4 associated with the partition (2^2). Thus, equation (85) is given by

$$(21^2)_{Sp_4,\beta}(\mathcal{E})\circ(2)_{\widetilde{Sl}_2,\gamma}(\sigma(\mathcal{E}))=(2^2)_{Sp_4,\beta,\gamma}(\mathcal{E}) \tag{88}$$

It follows from [G1] that the unipotent orbit (2^2) is associated with elements $\beta,\gamma \in (F^*)^2\backslash F^*$. The above identity is all we can say when β and γ are in general position. However, it is of interest to notice that when $\beta\gamma = -\epsilon^2$ for some $\epsilon \in F^*$ then this identity can be written in a different form. Indeed, when $\beta\gamma = -\epsilon^2$, one can find an element in $Sp_4(F)$, which depends on β and γ, such that the above integral is equal to

$$\int\limits_{(F\backslash \mathbf{A})^3} E_{\beta,\gamma}\begin{pmatrix} 1 & & x & y \\ & 1 & z & x \\ & & 1 & \\ & & & 1 \end{pmatrix}\psi(x)dxdydz$$

Here $E_{\beta,\gamma}$ is the right translation of the vector E by this discrete element. See [G-R-S2] page 880 for some details. Let w denote the Weyl element of Sp_4 defined by $w = e_{1,1} + e_{2,3} - e_{3,2} + e_{4,4}$. Here $e_{i,j}$ is the matrix of size four which has a one at the (i,j) entry, and zero otherwise. Since $w \in Sp_4(F)$, then $E(g) = E(wg)$. Hence, the above integral is equal to

$$\int\limits_{(F\backslash \mathbf{A})^3} E_{\beta,\gamma}\left(\begin{pmatrix} 1 & x & & y \\ & 1 & & \\ & z & 1 & -x \\ & & & 1 \end{pmatrix} w\right)\psi(x)dxdydz$$

Performing some Fourier expansions, one can show that the above integral is equal to

$$\int\limits_{\mathbf{A}}\sum_{\delta\in F^*}\int\limits_{R(F)\backslash R(\mathbf{A})} E_{\beta,\gamma}(rt(z)w)\psi_{R,\delta}(r)drdz + \int\limits_{\mathbf{A}}\int\limits_{F\backslash \mathbf{A}} E_{\beta,\gamma}^{U(P)}(m(x)t(z)w)\psi(x)dxdz$$

Here $t(z) = I_4 + ze_{3,2}$ and $m(x) = I_4 + x(e_{1,2} - e_{3,4})$. Also, the group R is the maximal unipotent subgroup of Sp_4, and $\psi_{R,\delta}$ is the Whittaker character of R defined as follows. Write $r \in R$ as $r = x_1(r_1)x_2(r_2)r'$. Here $x_1(r_1) = I_4 + r_1(e_{1,2} - e_{3,4})$ and $x_1(r_2) = I_4 + r_2e_{2,3}$. Then we define $\psi_{R,\delta}(r) = \psi(r_1 + \delta r_2)$. Finally, the group P is the maximal parabolic subgroup of Sp_4 whose Levi part is GL_2, and we denote by $U(P)$ its unipotent radical.

Ignoring the integration over the z variable, then in the notation of (85), when $\beta\gamma = -\epsilon^2$, this integral identity is given by

$$(21^2)_{Sp_4,\beta}(\mathcal{E}) \circ (2)_{\widetilde{Sl}_2,\gamma}(\sigma(\mathcal{E})) = \sum_{\delta\in F^*}(4)_{Sp_4,\delta}(\mathcal{E}) + \mathcal{CT}_{Sp_4,P}[(2)_{GL_2}(\mathcal{E}^{U(P)})] \tag{89}$$

We conclude that for some choice of unipotent elements u_0 and v_0, there is more than one way to write the identity (85). Experience indicate the following. There is a general expression for identity (85) which holds for all values of u_0 and v_0, and all representations $\mathcal{E}$. However, in some cases, there is a closed condition on u_0 and v_0 which will yield another identity. This is important once we specify the representation $\mathcal{E}$.

As an example to this phenomena, consider the group Sp_4, and the above two identities (88) and (89). Let τ denote an irreducible cuspidal representation of $GL_2(\mathbf{A})$ with a trivial

central character, and such that $L(\tau,1/2)\neq 0$. Let $E_\tau(g,s)$ denote the Eisenstein series of $Sp_4(\mathbf{A})$ associated with the induced representation $Ind_{P(\mathbf{A})}^{Sp_4(\mathbf{A})}\tau\delta_P^s$. The group P was defined right before identity (89). From the assumptions on τ it follows that this Eisenstein series has a simple pole at $s=2/3$, and we denote by $\mathcal{E}_\tau$ the residue representation at that point. In [G-R-S2] integral (86) was used to construct the descent map from τ to an irreducible cuspidal representation of $\widetilde{SL}_2(\mathbf{A})$. The proof of the nonvanishing of the descent used identity (89). Indeed, it is proved in [G-R-S2] that the first summand on the right hand side of (89) is zero and the second term is not. From this it was proved in [G-R-S2] that the descent given by integral (86) is not zero.

We may also consider the descent construction given by (86) where we take $\mathcal{E}$ to be a non-generic cuspidal representation of $Sp_4(\mathbf{A})$. In this case all constant terms are zero. Since $\mathcal{E}$ is not generic, we obtain for such representations that the right hand side of (89) is zero for all choice of data. Thus, equation (89) cannot be used in this case. Nevertheless, we can use equation (88) to deduce the nonvanishing of integral (86). Indeed, it is not hard to show that given any automorphic representation of $Sp_4(\mathbf{A})$, there exist β and γ as above, such that integral (87) is not zero for some choice of data.

Going back to the general case, one looks for a way to produce expansions of the form (85). To do that we will use the following approach. As in [C-M], to any unipotent orbit, one attaches a one dimensional torus in the group G in question. (The notations we use are as in [G1]). For example, the group Sp_4 has three nontrivial unipotent orbits. They are $(4),(2^2)$ and (21^2). The corresponding one dimensional tori are $h_{(4)}(t)=\text{diag }(t^3,t,t^{-1},t^{-3})$; $h_{(2^2)}(t)=$ $\text{diag }(t,t,t^{-1},t^{-1})$ and $h_{(21^2)}(t)=\text{diag }(t,1,1,t^{-1})$.

Suppose that we start with a unipotent orbit $\mathcal{O}_G$, and let $\psi_{U(\mathcal{O}_G),u_0}$ be a character of the unipotent group $U(\mathcal{O}_G)$. Let H be as defined right before equation (84), and suppose that $\mathcal{O}_H$ is a unipotent orbit of H. See (84) for notations. Let $h_{\mathcal{O}_G}(t)$ denote the one dimensional torus of G attached to $\mathcal{O}_G$, and let $h_{\mathcal{O}_H}(t)$ denote the one dimensional torus of H attached to $\mathcal{O}_H$. We view $h_{\mathcal{O}_H}(t)$ as a sub torus of G via the embedding of H in G. Thus, the product $h(t)=h_{\mathcal{O}_G}(t)h_{\mathcal{O}_H}(t)$ is a well defined one dimensional torus of G. Assume that there is a unipotent orbit $\mathcal{O}'_G$ of G such that $h(t)$ is conjugated by a certain Weyl element to the torus $h_{\mathcal{O}'_G}(t)$. Conjugating in (84) the argument of the function E by this Weyl element, will transform the integral (84) into an integral over a unipotent subgroup of $U(\mathcal{O}'_G)$. Then, using some Fourier expansions together with possible other conjugations, one can produce a formula of the type (85). At this point, we dont know of a general method that will predict the unipotent orbits and the constant terms which appear in equation (85). As can be seen from (88) and (89), the decomposition can be different if we vary the elements u_0 and v_0.

As an example to the above argument, consider the above composition $(21^2)\circ(2)$ in Sp_4. Here, to simplify the notations, we omitted several of them. We have $h_{(21^2)}(t) = \text{diag }(t,1,1,t^{-1})$. The torus element which corresponds to the partition (2) in SL_2 is diag (t,t^{-1}). When embedded into Sp_4, this torus corresponds to $h(t) = \text{diag }(1,t,t^{-1},1)$. Thus we obtain $h_{(21^2)}(t)h(t) = \text{diag }(t,t,t^{-1},t^{-1})$ which is equal to $h_{(2^2)}(t)$. Hence we dont need any conjugation here. The equation that we get is then $(21^2)\circ(2) = (2^2)$. But one has to remember that in certain closed conditions on the characters, one can derive another identity for this composition.

As an another example, consider the product $(2^3)\circ(3)$ in Sp_6. It follows from [C-M] that the reductive group in the stabilizer of the unipotent orbit (2^3) in Sp_6 is the group SO_3. Since we consider only the split stabilizer, the group SO_3 contains a one dimensional unipotent subgroup, and if we compute its Whittaker coefficient, we are considering the unipotent orbit (3). Thus, the composition $(2^3)\circ(3)$ corresponds to the integral

$$\int\limits_{F\backslash \mathbf{A}}\ \int\limits_{Mat^0_{3\times3}(F)\backslash Mat^0_{3\times3}(\mathbf{A})} E\left[\begin{pmatrix} I_3 & X \\ & I_3\end{pmatrix}\begin{pmatrix} m(y) & \\ & m(y)^*\end{pmatrix}\right]\psi_1(X)\psi_2(y)dXdy \tag{90}$$

Here $Mat^0_{3\times3} = \{r\in Mat_{3\times3} : J_3r + r^tJ_3 = 0\}$ and

$$J_3 = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \qquad m(y) = \begin{pmatrix} 1 & y & * \\ & 1 & -y \\ & & 1\end{pmatrix}$$

The star indicates that the matrix is in SO_3. Also, we define $\psi_1(X) = \psi(x_{1,1}+x_{2,2})$, and $\psi(m_2(y)) = \psi(y)$. We remark that this is not the general character which is associated to this unipotent orbit, such that the stabilizer is the split SO_3. The general one is given by $X\mapsto \psi(x_{1,1}+\beta x_{2,2})$ where $\beta\in (F^*)^2\backslash F^*$. However, the stabilizer in each case is the same up to an outer conjugation, and hence the formulas are the same.

Before conjugation, it will be convenient to transfer integral (90) to another integral using the process of exchanging roots. See subsection 2.2.2. In the above integral we replace the one dimensional unipotent group $I_6 + x_{3,1}e_{3,4}$ in the X variable by $I_6 + y_3(e_{1,3} - e_{4,6})$ and then $I_6 + x_{2,1}(e_{2,4} + e_{3,5})$ in the X variable by $I_6 + y_2(e_{2,3} - e_{4,5})$. More precisely, we expand integral (90) along the unipotent group $I_6 + y_2(e_{2,3} - e_{4,5}) + y_3(e_{1,3} - e_{4,6})$. Then we conjugate by a suitable discrete element in $Sp_6(F)$ and then perform a collapsing of summation with

integration. Thus, integral (90) is equal to

$$\int\limits_{\mathbf{A}^2}\int\limits_{(F\backslash\mathbf{A})^7} E\left[\begin{pmatrix}1 & y_1 & y_2 & & & \\ & 1 & y_3 & & & \\ & & 1 & & & \\ & & & 1 & * & * \\ & & & & 1 & * \\ & & & & & 1\end{pmatrix}\begin{pmatrix}1 & & & x_1 & x_2 & x_3 \\ & 1 & & & x_4 & x_2 \\ & & 1 & & & x_1 \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1\end{pmatrix} l(z_1,z_2)\right]\psi(y_1+x_1+x_4)d(...)$$

Here $l(z_1,z_2)=I_6+z_1(e_{2,4}+e_{3,5})+z_2e_{3,4}$.

It follows from [C-M] that $h_{(2^3)}(t) = \text{diag }(t,t,t,t^{-1},t^{-1},t^{-1})$. We also have $h_{(3)}(t) = \text{diag }(t^2,1,t^{-2},t^2,1,t^{-2})$, where the last torus element is the corresponding torus element in SO_3 as embedded in Sp_6. Thus, the product of these two tori is given by $h(t) = \text{diag }(t^3,t,t^{-1},t,t^{-1},t^{-3})$. Consider the Weyl element w of Sp_6 given by $w_{1,1}=w_{2,4}=w_{3,2}=w_{4,5}=w_{6,6}=1$ and $w_{5,3}=-1$. Then $wh(t)w^{-1}=\text{diag }(t^3,t,t,t^{-1},t^{-1},t^{-3})$, and this torus is equal to $h_{(42)}(t)$.

Since $w\in Sp_6(F)$, the above integral is equal to

$$\int\limits_{\mathbf{A}^2}\int\limits_{(F\backslash\mathbf{A})^7} E(m(y_i,x_j)wl(z_1,z_2))\psi(y_1+y_2+x_5)d(...)$$

where

$$m(y_i,x_j)=\begin{pmatrix}1 & y_1 & y_2 & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & -y_2 \\ & & & & 1 & -y_1 \\ & & & & & 1\end{pmatrix}\begin{pmatrix}1 & & & x_1 & x_2 & x_3 \\ & 1 & & x_4 & & x_2 \\ & & 1 & x_5 & x_4 & x_1 \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1\end{pmatrix}$$

Next we expand the above integral along the unipotent group $l(x_6)=I_6+x_6e_{2,5}$. We obtain

$$\int\limits_{\mathbf{A}^2}\sum_{\beta\in F}\int\limits_{(F\backslash\mathbf{A})^8} E(m(y_i,x_j)l(x_6)wl(z_1,z_2))\psi(y_1+y_2+x_5+\beta x_6)d(...)$$

Partition the sum in the above integral into two summands. First, consider the case when $\beta\in F^*$. In this case, it follows from [G1] that for each β, the corresponding Fourier coefficient is associated with the unipotent orbit (42). When $\beta=0$ we can further manipulate the integral. Indeed, conjugation by $s=I_6-e_{2,3}+e_{4,5}\in Sp_6(F)$ we obtain the integral

$$\int\limits_{\mathbf{A}^2}\int\limits_{(F\backslash\mathbf{A})^8} E(m(y_i,x_j)l(x_6)swl(z_1,z_2))\psi(y_2+x_5)d(...)$$

Conjugating by a certain Weyl element, and using further Fourier expansions, we can show that this integral is a sum of two terms. The first corresponds to the Whittaker coefficient

of the function E, and the second to a certain constant term. We omit the details. Thus we obtain the formula

$$(2^3)_{Sp_6} \circ (3)_{SO_3} = \sum_{\beta \in F^*} (42)_{Sp_6,\beta} + \sum_{\alpha \in F^*} (6)_{Sp_6,\alpha} + \mathcal{CT}_{Sp_6,P}[(4)_{Sp_4}] \tag{91}$$

We close this subsection with two remarks. The first one is concerning the dimensions of the orbits and the representations in question. Recall that in considering possible descent constructions we required a certain dimension formula to hold. This was equation (81), given by

$$\dim\, \mathcal{E} = \frac{1}{2}\dim\, \mathcal{O} + \dim\, \sigma$$

For the descent to be nonzero, the representation $\mathcal{E}$ should support a nontrivial Fourier coefficient with respect to the unipotent orbit occurring in the left hand side of (85). However, the dimension of the unipotent integration which occurs in integral (84) is $\dim\, U(\mathcal{O}_G) + \dim\, V(\mathcal{O}_H)$. By definition this number is equal to $\frac{1}{2}\dim\, \mathcal{O}_G + \dim\, \sigma$. This motivates to look for those representations $\mathcal{E}$ of $G(\mathbf{A})$ which satisfies the following. First, that $\mathcal{O}_G(\mathcal{E})$ is equal to a unipotent orbit corresponding to one of the summands occurring on the right hand side of (85). Second, we require that the representation does not support any Fourier coefficient which corresponds to any other term which occurs on the right hand side of (85).

To illustrate this consider the above two examples in the symplectic group. First, the Sp_4 case. Notice that $\dim\, (21^2)_{Sp_4} = 4; \dim\, (2^2)_{Sp_4} = 6$ and $\dim\, (2)_{SL_2} = 2$. Hence $\dim\, \mathcal{E} = \frac{1}{2}\dim\, (21^2)_{Sp_4} + \dim\, \sigma = 2 + 1 = 3$. Hence we look for those representations such that $\mathcal{O}_{Sp_4}(\mathcal{E}) = (2^2)$. This can work if we use equation (88). However, if we want to use equation (89), then we need to assume also that the representation $\mathcal{E}$ is not generic. In the Sp_6 case the situation is as follows. The sum of the half of the dimensions of the unipotent orbits which occur in the left hand side of (91) is $6 + 1 = 7$. However, half of the dimension of $(42)_{Sp_6}$ is eight and of $(6)_{Sp_6}$ is nine. Hence the only way to get a term on the right hand side of (91) whose half of the dimension is seven is to look for a representation $\mathcal{E}$ of $Sp_6(\mathbf{A})$ such that it has no nonzero Fourier coefficient associated with any representative of the orbits (42) and (6), such that the integral associated with $\mathcal{CT}_{Sp_6,P}[(4)_{Sp_4}]$ is not zero. It is not clear if such a representation exists.

The second remark concerns the cuspidality of the descent. The goal is to compute integral (84) where the group $V(\mathcal{O}_H)$ is a constant term, and the character $\psi_{V(\mathcal{O}_H),v_0}$ is the trivial character. Then, instead of the left hand side of (85), one should compute $\mathcal{O}_{G,u_0}(\mathcal{E}) \circ \mathcal{CT}(\sigma(\mathcal{E}))$. By that we mean that one should express this convolution as a sum of unipotent orbits of G and certain constant term of the representation involved. Experience indicates

that at least one of the terms will involve a constant term of the group G, but so far we cannot indicate which one, and we also cannot predict in general the other terms.

4.2.2. **Unipotent Orbits and Torus Elements for F_4.** In this subsection we determine the one dimensional torus element attached to a given unipotent orbit of F_4.

Recall from subsection 2.2 that we can partition the set of roots in the group U_Δ as follows. As in that subsection, we will say that a root α is in the unipotent group U_Δ, if the one parameter unipotent subgroup $\{x_\alpha(r)\}$ is a subgroup of U_Δ. For all $n \geq 1$, we defined $U'_\Delta(n) = \{\alpha = \sum_{i=1}^4 n_i\alpha_i \in U_\Delta : \sum_{i=1}^4 \epsilon_i n_i = n\}$. We can extend this notation and write $U_\Delta(0)$ for all positive roots in the Levi part of the parabolic group P_Δ. Let $h_\mathcal{O}(t)$ denote the one dimensional torus of F_4 with the property that for all $\alpha \in U'_\Delta(n)$ we have

$$h_\mathcal{O}(t)x_\alpha(r)h_\mathcal{O}(t)^{-1} = x_\alpha(t^n r) \tag{92}$$

It follows from the Bala-Carter theory that such a torus exists. For details in the classical groups see [C-M]. To compute this torus in F_4, let $h_\mathcal{O}(t) = h(t^{r_1}, t^{r_2}, t^{r_3}, t^{r_4})$. Then, given a root $\alpha \in U'_\Delta(n)$, equation (92) reduces to the equation $\sum_{i=1}^4 r_i < \alpha, \alpha_i >= n$. Here $< \alpha, \alpha_i >$ is the inner product between the root α and the simple root α_i. It is easy to solve these equations in general, and the solution can be derived form the following 4 identities

$$r_1 = \mathcal{G}_\mathcal{O}(2342); \quad r_2 = r_1 + 2r_4 - \mathcal{G}_\mathcal{O}(1122); \quad r_3 = \frac{1}{2}(r_2 + \mathcal{G}_\mathcal{O}(1242)); \quad r_4 = \mathcal{G}_\mathcal{O}(1232)$$

Here, for a positive root $\alpha \in U'_\Delta(n)$, we define $\mathcal{G}_\mathcal{O}(\alpha)$ as follows. Let $\alpha = \sum n_i\alpha_i$ and suppose that the diagram of $\mathcal{O}$ is given by

$$\overset{\epsilon_1}{\alpha_1} ----\overset{\epsilon_2}{\alpha_2} ==>== \overset{\epsilon_3}{\alpha_3} ----\overset{\epsilon_4}{\alpha_4}$$

Then we define $\mathcal{G}_\mathcal{O}(\alpha) = \sum \epsilon_i n_i$.

As an example consider the unipotent orbit $\mathcal{O} = B_2$. Its diagram is

$$\overset{2}{0} ----0 ==>== 0 ----\overset{1}{0}$$

Hence $\mathcal{G}_\mathcal{O}(2342) = 2\cdot 2 + 1 \cdot 2 = 6$ and $\mathcal{G}_\mathcal{O}(1122) = \mathcal{G}_\mathcal{O}(1242) = \mathcal{G}_\mathcal{O}(1232) = 4$. Thus, $r_1 = 6; r_2 = 10;\ r_3 = 7;\ r_4 = 4$ and $h_{B_2}(t) = h(t^6, t^{10}, t^7, t^4)$.

We list the set of all 15 tori elements in F_4:

1) $h_{A_1}(t) = h(t^2, t^3, t^2, t)$.

2) $h_{\widetilde{A}_1}(t) = h(t^2, t^4, t^3, t^2)$.

3) $h_{A_1+\widetilde{A}_1}(t) = h(t^3, t^6, t^4, t^2)$.

4) $h_{A_2}(t) = h(t^4, t^6, t^4, t^2)$.

5) $h_{\widetilde{A}_2}(t) = h(t^4, t^8, t^6, t^4)$.

6) $h_{A_2+\widetilde{A}_1}(t) = h(t^4, t^8, t^6, t^3)$.

7) $h_{B_2}(t) = h(t^6, t^{10}, t^7, t^4)$.

8) $h_{\widetilde{A}_2+A_1}(t) = h(t^5, t^{10}, t^7, t^4)$.
9) $h_{C_3(a_1)}(t) = h(t^6, t^{11}, t^8, t^4)$.
10) $h_{F_4(a_3)}(t) = h(t^6, t^{12}, t^8, t^4)$.
11) $h_{B_3}(t) = h(t^{10}, t^{18}, t^{12}, t^6)$.
12) $h_{C_3}(t) = h(t^{10}, t^{19}, t^{14}, t^8)$.
13) $h_{F_4(a_2)}(t) = h(t^{10}, t^{20}, t^{14}, t^8)$.
14) $h_{F_4(a_1)}(t) = h(t^{14}, t^{26}, t^{18}, t^{10})$.
15) $h_{F_4}(t) = h(t^{22}, t^{42}, t^{30}, t^{16})$.

4.3. **Conditions for Cuspidality and Nonvanishing of the Descents.** In this subsection we shall work out the global setup in some of the cases mentioned in subsection 4.1. The choice is partly random and partly motivated by considering examples we think to be of some interest. More precisely, our concern is to give in each case conditions when the descent is cuspidal and when it is not zero. To do that we compute integral (84) in the case when it is a Fourier coefficient corresponding to the relevant unipotent orbit, or when the integration over V represents a constant term along a certain unipotent radical of a maximal parabolic subgroup of H. Therefore, the precise starting integral, whether it is integral (73) or (74) will not be important to us, hence we ignore it. For our goal, it is enough to indicate in each case the group U'_Δ and the character ψ_{U,u_Δ}. See integral (82) for notations. We will express the answer in terms of the notations used in equation (85).

Since the question of cuspidality and of the nonvanishing is a statement of certain integral being zero for all choice of data or not, it will be convenient in many case to ignore adelic integration which occurs during the computations. Indeed, when performing root exchange, as explained in subsection 2.2.2, we relate a certain Fourier coefficient with a certain integral which involves adelic integration. However, in all cases one can easily prove that one integral is zero for all choice of data if and only if the other one is zero for all choice of data. For our purposes that is enough. In some cases we will still write the equation (85), but we mean that the left hand side is zero for all choice of data if and only if each term on the right hand side is zero for all choice of data.

4.3.1. **The Unipotent Orbit** C_3**.** The construction of the unipotent group U'_Δ and the characters ψ_{U,u_Δ} were described in Section 2. In this case the group U'_Δ is as follows. Let U denote the unipotent radical of the parabolic subgroup of F_4 whose Levi part contains the SL_2 generated by $\{x_{\pm 0100}\}$. Thus, $U = U_{\alpha_2}$ and $\dim U = 23$. Let U'_Δ denote the subgroup of U which consists of all one dimensional unipotent subgroup $\{x_\alpha(r)\}$ where α is a root in U which does not include the roots (0010) and (0110). Thus $\dim U'_\Delta = 21$. We define the character ψ_{U,u_Δ} as follows. For $u = x_{0001}(r_1)x_{1110}(r_2)x_{0120}(r_3)u'$ define $\psi_{U,u_\Delta}(u) = \psi(r_1 + r_2 + r_3)$. (

See subsection 2.1 for notations). Thus, the group $SL_2 =< x_{\pm 0100}(m) >$ is in the stabilizer of ψ_{U,u_Δ}. To simplify notations, we shall denote the group U'_Δ by V, and the character ψ_{U,u_Δ} by ψ_V.

This diagram associated with this unipotent orbit contains nodes which are labelled with ones. Hence the construction we use is (74). Let σ denote the representation defined on $SL_2(\mathbf{A})$ which is obtained by integral (74). This copy of SL_2 splits under the double cover of the relevant symplectic group. We look for conditions when σ is cuspidal and when it is not zero. We start with the nonvanishing. Thus, we compute

$$\int\limits_{F\backslash \mathbf{A}}\int\limits_{V(F)\backslash V(\mathbf{A})} E(vx_{0100}(r))\psi_V(v)\psi(ar)drdv$$

where $a \in F^*$. In the notations of integral (84), we have $V(\mathcal{O}_H) = \{x_{0100}(r)\}$ and $V(\mathcal{O}_G) = V$.

Notice that for this orbit we have $h_{C_3}(t) = h(t^{10}, t^{19}, t^{14}, t^8)$. Also, from the embedding in F_4 of the group of type A_1, which is inside the stabilizer of this orbit inside F_4, we deduce that its maximal torus is $h(1, t, 1, 1)$. Thus the product of the two tori gives $h_{C_3}(t)h(1, t, 1, 1) = h(t^{10}, t^{20}, t^{14}, t^8) = h_{F_4(a_2)}(t)$. Thus we expect to obtain the orbit $F_4(a_2)$ in the expansion, and we dont need any conjugation by some Weyl elements.

We expand along the unipotent group $\{x_{0110}(m)\}$. The above integral is equal to

$$\sum_{\gamma\in F}\int\limits_{(F\backslash \mathbf{A})^2}\int\limits_{V(F)\backslash V(\mathbf{A})} E(vx_{0110}(m)x_{0100}(r))\psi_V(v)\psi(ar+\gamma m)drdmdv \tag{93}$$

Since the function E is automorphic, then for all $\gamma \in F$ we have $E(h) = E(x_{1000}(\gamma)h)$. Using that, and conjugating $x_{1000}(\gamma)$ to the right, integral (93) is equal to

$$\int\limits_{\mathbf{A}}\int\limits_{V_1(F)\backslash V_1(\mathbf{A})} E(v_1x_{1000}(r))\psi_{V_1,a}(v_1)drdv_1 \tag{94}$$

In the derivation of the above integral we also collapsed the summation over γ with the suitable integration. Here V_1 is the unipotent radical of the parabolic subgroup of F_4 whose Levi part contains $SL_2 \times SL_2$ which is generated by $< x_{\pm 1000}(r), x_{\pm 0010}(r) >$. In other words, $V_1 = U_{\alpha_1,\alpha_3}$, and hence $\dim V_1 = 22$. The character $\psi_{V_1,a}$ is defined as follows. For $v_1 = x_{0001}(r_1)x_{1110}(r_2)x_{0120}(r_3)x_{0100}(r_4)v'_1$ let $\psi_{V_1,a}(v_1) = \psi(r_1+r_2+r_3+ar_4)$. It follows from Section 2 that the Fourier coefficient along V_1 corresponds to the unipotent orbit $F_4(a_2)$. Arguing as in [Ga-S], integral (94) is nonzero for some choice of data if and only if the Fourier coefficient along V_1 is not zero for some choice of data. From this we conclude that the representation σ is not zero if and only if the representation $\mathcal{E}$ has a nonzero Fourier

coefficient corresponding to the unipotent orbit $F_4(a_2)$, which corresponds to the character $\psi_{V_1,a}$.

To study when σ is cuspidal, we consider the constant term along the unipotent radical of the Borel subgroup of the group SL_2. Thus, we need to compute the integral

$$\int_{F\backslash\mathbf{A}}\int_{V(F)\backslash V(\mathbf{A})} E(vx_{0100}(r))\psi_V(v)drdv$$

Let V_1 denote the unipotent group generated by V and $\{x_{0100}(r)\}$. Since E is automorphic, we obtain $E(h) = E(w[3124321]h)$. Conjugating the above integral by this Weyl element, the above integral is equal to

$$\int_{L(F)\backslash L(\mathbf{A})}\int_{U_2(F)\backslash U_2(\mathbf{A})}\int_{U_1(F)\backslash U_1(\mathbf{A})} E(u_1u_2lw_0)\psi_{U_1}(u_1)du_1du_2dl \tag{95}$$

Here U_1 is the maximal unipotent subgroup of Sp_6 as embedded inside a Levi part of a maximal parabolic subgroup of F_4. The character ψ_{U_1} is the Whittaker character of U_1. In other words, $\psi_{U_1}(u_1) = \psi(x_{0100}(r_1)x_{0010}(r_2)x_{0001}(r_3)u_1') = \psi(r_1 + r_2 + r_3)$. The group U_2 is the unipotent subgroup of F_4 generated by all $\{x_\alpha(r)\}$ where α is a root in the set $\{(1122); (1221); (1231); (1222); (1232); (1242); (1342); (2342)\}$. Thus $\dim U_2 = 8$. The unipotent group L is generated by all one parameter unipotent subgroups $\{x_\alpha(r)\}$ where α is a root in the set $\{-(1000); -(1100); -(1110); -(1120); -(1111)\}$. The dimension of L is 5. Finally, we denoted $w_0 = w[3124321]$.

Next we consider a series of root exchange in integral (95). (See Section 2). We first expand along $\{x_{1220}(r)\}$. Thus integral (95) is equal to

$$\int_{L(F)\backslash L(\mathbf{A})}\sum_{\gamma\in F}\int_{F\backslash\mathbf{A}}\int_{U_2(F)\backslash U_2(\mathbf{A})}\int_{U_1(F)\backslash U_1(\mathbf{A})} E(x_{1220}(r)u_1u_2lw_0)\psi_{U_1}(u_1)\psi(\gamma r)du_1du_2drdl \tag{96}$$

Using the fact that E is automorphic we have $E(h) = E(x_{-(1120)}(\gamma)h)$. Conjugating this element to the right, changing variables, and collapsing summation with integration, integral (96) is equal to

$$\int_{\mathbf{A}}\int_{L_1(F)\backslash L_1(\mathbf{A})}\int_{U_3(F)\backslash U_3(\mathbf{A})}\int_{U_1(F)\backslash U_1(\mathbf{A})} E(u_1u_3l_1x_{-(1120)}(m)w_0)\psi_{U_1}(u_1)du_1du_3dl_1dm \tag{97}$$

Here U_3 corresponds to the unipotent group generated by all $\{x_\alpha(r)\}$ where α is a root in the set

$$\{(1220; (1122); (1221); (1222); (1231); (1232); (1242); (1342); (2342)\}$$

The group L_1 is generated by all one dimensional unipotent subgroups $\{x_\alpha(r)\}$ where α is a root in the set $\{-(1000); -(1100); -(1110); -(1111)\}$. We repeat this process two more

times. First we expand along the unipotent group $\{x_{1121}(r)\}$ and use for that the unipotent group $\{x_{-(1111)}(m)\}$. Then we expand along the group $\{x_{1120}(r)\}$ and use the group $\{x_{-(1110)}(m)\}$. Thus, integral (97) is equal to

$$(98)\qquad \int\limits_{Z(\mathbf{A})}\int\limits_{L_2(F)\backslash L_2(\mathbf{A})}\int\limits_{U_4(F)\backslash U_4(\mathbf{A})}\int\limits_{U_1(F)\backslash U_1(\mathbf{A})} E(u_1u_4l_2zw_0)\psi_{U_1}(u_1)du_1du_4dl_2dz$$

Here, the group U_4 is generated by all $\{x_\alpha(r)\}$ such that α is a root in the set

$$\{(1120);(1121);(1220);(1122);(1221);(1222);(1231);(1232);(1242);(1342);(2342)\}$$

The group L_2 is generated by all $\{x_\alpha(r)\}$ such that α is a root in the set $\{-(1000);-(1100)\}$, and Z is generated by all $\{x_\alpha(r)\}$ such that α is a root in the set $\{-(1110);-(1111);-(1120)\}$. Arguing as in [Ga-S], integral (98) is zero for all choice of data, if and only if the integral

$$(99)\qquad \int\limits_{L_2(F)\backslash L_2(\mathbf{A})}\int\limits_{U_4(F)\backslash U_4(\mathbf{A})}\int\limits_{U_1(F)\backslash U_1(\mathbf{A})} E(u_1u_4l_2)\psi_{U_1}(u_1)du_1du_4dl_2$$

is zero for all choice of data. Next we expand integral (99) along the unipotent subgroup $\{x_{1111}(r)\}$. Thus, integral (99) is a sum of two integral. The first is the contribution to (99) from the nontrivial orbit. In this case, after conjugation by the Weyl element $w[21]$, it follows from the description of the unipotent orbits given in Section 2, that the expansion obtained is a Fourier coefficient which corresponds to the unipotent orbit $F_4(a_1)$. We denote this integral by I_1. The second integral, denoted by I_2, is the contribution from the constant term. In this case we proceed as above. We expand along the unipotent group $\{x_{1110}(r)\}$ and use the unipotent group $\{x_{-(1100)}(m)\}$, and then expand along $\{x_{1100}(r)\}$ and use the group $\{x_{-(1000)}(m)\}$. Thus, integral I_2 is zero for all choice of data if and only if the integral

$$(100)\qquad \int\limits_{U_5(F)\backslash U_5(\mathbf{A})}\int\limits_{U_1(F)\backslash U_1(\mathbf{A})} E(u_1u_4l_2)\psi_{U_1}(u_1)du_1du_5$$

is zero for all choice of data. Here U_5 is the unipotent group which is generated by U_4 and $\{x_{1100}(r_1)x_{1110}(r_2)\}$. Finally, we expand (100) along the unipotent group $\{x_{1000}(r)\}$. There are two cases. The first, corresponds to the nontrivial orbit, produce a Fourier coefficient which is associated with the unipotent orbit F_4. The other case, which corresponds to the constant term, contributes the integral

$$(101)\qquad \int\limits_{U_1(F)\backslash U_1(\mathbf{A})} E^{U(C_3)}(u_1)\psi_{U_1}(u_1)du_1$$

Here $U(C_3) = U_{\alpha_2,\alpha_3,\alpha_4}$, and $E^{U(C_3)}$ denotes the constant term of E along $U(C_3)$. We summarize,

Proposition 23. *Let $\mathcal{E}$ denote an automorphic representation of $F_4(\mathbf{A})$ such that:*
1) *The representation $\mathcal{E}$ has no nonzero Fourier coefficients associated with the unipotent orbits $F_4(a_1)$ and F_4. Also, integral* (101) *is zero for all choice of data.*
2) *There exists an $a \in F^*$, such that the representation $\mathcal{E}$ has a nonzero Fourier coefficient associated with the unipotent orbit $F_4(a_2)$ which is given by integral* (94)*.*

Then the representation σ is a nonzero cuspidal representation of the group $SL_2(\mathbf{A})$.

It follows from the above that identity (85) can be described in this case by

$$C_3(\mathcal{E}) \circ (2)_a = F_4(a_2)_a$$

Here $a \in (F^*)^2 \backslash F^*$.

4.3.2. **The Unipotent Orbit B_3.** We consider the descent construction which is obtained from the unipotent orbit B_3. In this case, the group U'_Δ, and the character ψ_{U,u_Δ} given in integral (73) are as follows. The group U'_Δ is the unipotent radical of the parabolic subgroup of F_4 whose Levi part contains the group SL_3 generated by $< x_{\pm(0010)}(r), x_{\pm(0001)}(r) >$. Thus, $U'_\Delta = U_{\alpha_3,\alpha_4}$. To define the character ψ_{U,u_Δ}, write $u = x_{0111}(r_1)x_{0120}(r_2)x_{1000}(r_3)u'$. Then $\psi_{U,u_\Delta}(u) = \psi(r_1 + r_2 + r_3)$. The stabilizer of ψ_{U,u_Δ} in the above copy of SL_3 is the group SO_3. For short we write V for U'_Δ and ψ_V for ψ_{U,u_Δ}. Thus, integral (73) produces an automorphic representation σ on $SO_3(\mathbf{A})$. We have $h_{B_3}(t) = h(t^{10}, t^{18}, t^{12}, t^6)$. The maximal torus of SO_3 is given by $h(1, 1, m, m)$ where $m \in F^*$. Hence, the maximal torus of SL_2 as embedded in SO_3 is given by $h(1, 1, t^2, t^2)$. We have $h_{B_3}(t)h(1, 1, t^2, t^2) = h(t^{10}, t^{18}, t^{14}, t^8)$. Conjugating this torus by w_2 we obtain $h_{F_4(a_2)}(t) = h(t^{10}, t^{20}, t^{14}, t^8)$. Thus we expect to get the unipotent orbit $F_4(a_2)$, after a suitable conjugation by a Weyl element. The maximal unipotent subgroup of SO_3 is embedded in F_4 as $j(r) = x_{0010}(r)x_{0001}(\eta_1 r)x_{0011}(\eta_2 r)$ where η_i are some fixed elements in F^* determined so that $j(r)$ stabilizes the character ψ_V.

The integral we need to compute is given by

$$\int\limits_{F\backslash\mathbf{A}} \int\limits_{V(F)\backslash V(\mathbf{A})} E(vj(r))\psi(ar)\psi_V(v)drdv \tag{102}$$

where $a = 0, 1$. Thus, if $a = 0$ we compute the constant term of σ, whereas if $a = 1$ we compute the Whittaker coefficient of σ.

In both cases we start with two root exchanges as explained in subsection 2.2.2. First we perform a Fourier expansion along the unipotent group $\{x_{0011}(m)\}$ and exchange it by $\{x_{0100}(l)\}$. Then we repeat this process with the roots (0001) and (0110). In the case when $a = 1$ we also exchange (1100) by $-(0100)$. Assume that $a = 1$. Then when we conjugate

the above integral by $w[32]$, we obtain as inner integration the integral

$$\int\limits_{U_{\alpha_1,\alpha_3}(F)\backslash U_{\alpha_1,\alpha_3}(\mathbf{A})} E(u)\psi_U(u)du \tag{103}$$

where ψ_U is defined as follows. Write $u \in U_{\alpha_1,\alpha_3}$ as $u = x_{0001}(r_1)x_{0100}(r_2)x_{0110}(r_3)x_{1120}(r_4)u'$. Then $\psi_U(u) = \psi(r_1 + r_2 + r_3 + r_4)$. It follows from subsection 2.2.1, that this Fourier coefficient corresponds to the unipotent orbit $F_4(a_2)$. Also, it is not hard to check that this integral is zero for all choice of data, if and only if integral (102) is zero for all choice of data.

The case when $a = 0$ is done in a similar way as in the case of the unipotent orbit C_3. After performing the above two root exchange, we conjugate the integral by $w_0 = w[432132]$, and we obtain that integral (102), with $a = 0$, is zero for all choice of data, if and only if the integral

$$\int\limits_{L(F)\backslash L(\mathbf{A})} \int\limits_{U_2(F)\backslash U_2(\mathbf{A})} \int\limits_{U_1(F)\backslash U_1(\mathbf{A})} E(u_1u_2lw_0)\psi_{U_1}(u_1)du_1du_2dl$$

is zero for all choice of data. Here U_1 is the maximal unipotent subgroup of $Spin_7$ which is embedded in F_4 as a Levi part of a maximal parabolic subgroup. The character ψ_{U_1} is the Whittaker character defined on U_1. Thus, if $u \in U_1$ is written as $u = x_{1000}(r_1)x_{0100}(r_2)x_{0010}(r_3)u'$, then $\psi_{U_1} = \psi(r_1 + r_2 + r_3)$. The group U_2 is generated by all $\{x_\alpha(r)\}$ where

$$\alpha \in \{(1111);\ (0121);\ (1121);\ (1221);\ (1231);\ (1232);\ (1242);\ (1342);\ (2342)\}$$

Finally, the group L is generated by all $\{x_\alpha(r)\}$ such that

$$\alpha \in \{-(1122);\ -(0122);\ -(0011);\ -(0001)\}$$

This integral is similar to the integral (95). We proceed in a similar way as in the case of the unipotent orbit C_3. Since the computations are similar, we shall omit them. To state the conditions we obtain, we consider the integral

$$\int\limits_{U_1(F)\backslash U_1(\mathbf{A})} E^{U(B_3)}(u_1)\psi_{U_1}(u_1)du_1 \tag{104}$$

Here $U(B_3) = U_{\alpha_1,\alpha_2,\alpha_3}$. We have,

Proposition 24. *Let $\mathcal{E}$ denote an automorphic representation of $F_4(\mathbf{A})$ such that:*
1) *The representation $\mathcal{E}$ has no nonzero Fourier coefficients associated with the unipotent orbits $F_4(a_1)$ and F_4. Also, integral* (104) *is zero for all choice of data.*
2) *The representation $\mathcal{E}$ has a nonzero Fourier coefficient associated with the unipotent orbit $F_4(a_2)$ which is given by integral* (103)*.*

Then the representation σ is a nonzero cuspidal representation of the group $SO_3(\mathbf{A})$.

As in the previous case, we can rephrase the nonvanishing computation in terms of identity (85). In this case we have

$$B_3(\mathcal{E}) \circ (3) = F_4(a_2)$$

4.3.3. **The Unipotent Orbit** $C_3(a_1)$**.** The diagram corresponding to this unipotent orbit contains nodes which are labelled one, and hence we use the integral (74) where the theta representation is defined on the double cover of $Sp_6(\mathbf{A})$. It follows from [C] that the stabilizer of this unipotent orbit is a group of type A_1. In this case it is the group $SL_2(\mathbf{A})$, and according to the choice of character $\psi_{U,\Delta}$ which will be specified below, we have $SL_2 =< x_{\pm(0100)} >$. From the embedding of this copy of SL_2 inside Sp_6, we deduce that the representation σ is defined over the double cover of SL_2.

Since our goal is to study the vanishing or nonvanishing of certain Fourier coefficients, it is enough to study integral (76). Thus we need to describe the group U'_Δ and the character ψ_{U,u_Δ} that we choose. From the description of the diagram associated with this unipotent orbit, it follows that $U_\Delta = U_{\alpha_1,\alpha_3}$. Let U'_Δ denote the subgroup of U_Δ which consists of all roots in U_Δ deleting the three roots (0010); (0011); (1000). Thus $\dim U'_\Delta = 19$. The character ψ_{U,u_Δ} is defined as follows. For $u' \in U'_\Delta$ write $u' = x_{0121}(r_1)x_{1110}(r_2)x_{1111}(r_3)u''$. Then $\psi_{U,u_\Delta}(v') = \psi(r_1 + r_2 + r_3)$. Denote $V = U_\Delta$ and $V' = U'_\Delta$. The one dimensional torus associated with this orbit is $h_{C_3(a_1)}(t) = h(t^6, t^{11}, t^8, t^4)$. The one dimensional torus of the copy of SL_2 which is the stabilizer of ψ_{U,u_Δ} is $h(1, t, 1, 1)$. If we multiply these two tori elements, we obtain $h_{F_4(a_3)}(t)$.

Thus, the integral we need to consider is given by

$$\int_{F\backslash \mathbf{A}} \int_{V'(F)\backslash V'(\mathbf{A})} E(v' x_{0100}(r))\psi_{U,u_\Delta}(v')\psi(\beta r)dv'dr \tag{105}$$

Here $\beta \in F$. We start with the case when $\beta \neq 0$. In this case, the above integral is equal to

$$\int_{U_{\alpha_2}(F)\backslash U_{\alpha_2}(\mathbf{A})} E(u)\psi_{U,u_\Delta}(u)du$$

where now the character ψ_{U,u_Δ} is a character of the group U_{α_2}, and is given as follows. Write $u \in U_{\alpha_2}$ as $u = x_{0121}(r_1)x_{1110}(r_2)x_{1111}(r_3)x_{0100}(r_4)u'$. Then $\psi_{U,u_\Delta}(u) = \psi(r_1 + r_2 + r_3 + \beta r_4)$. It follows from subsection 2.2.1 that this Fourier coefficient corresponds to the unipotent orbit $F_4(a_3)$. Indeed, in the notations of equation (9), the character ψ_{U,u_Δ} corresponds to the character $\psi_{U_\Delta,A,B}$ with

$$A = \begin{pmatrix} & 1 & \\ & & 1 \\ \beta & & \end{pmatrix} \qquad B = \begin{pmatrix} 1 & & & \\ 1 & & & \\ & & 1 & 1 \end{pmatrix}$$

Solving the equations given in (10), we obtain that the only solution is the trivial solution. Hence, the above character ψ_{U,u_Δ} is associated with the unipotent orbit $F_4(a_3)$. In the notations of equation (85) we proved

$$C_3(a_1)(\mathcal{E}) \circ (2)_\beta = F_4(a_3)_\beta$$

Next we consider the cuspidality of the lift. Thus, we need to compute integral (105) with $\beta = 0$. Let $w_0 = w[1234213]$. Conjugating by this element, integral (105) is equal to

$$\int\limits_{L(F)\backslash L(\mathbf{A})} \int\limits_{U_2(F)\backslash U_2(\mathbf{A})} \int\limits_{U_1(F)\backslash U_1(\mathbf{A})} E(u_1u_2lw_0)\psi_{U_1}(u_1)du_1du_2dl \tag{106}$$

Here U_1 is the unipotent subgroup of Sp_6 which is generated by all $\{x_\alpha(r)\}$ where $\alpha = \sum n_i\alpha_i$ such that $n_1 = 0$ and deleting the simple root α_2. Thus dim $U_1 = 8$. The character ψ_{U_1} is defined as follows. Write $u_1 = x_{0001}(r_1)x_{0110}(r_2)x_{0011}(r_3)u_1'$. Then $\psi_{U_1}(u_1) = \psi(r_1 + r_2 + r_3)$. We mention that this Fourier coefficient of the group corresponds to the unipotent orbit (42) in the group Sp_6. The group U_2 is generated by all $\{x_\alpha(r)\}$ such that

$$\alpha \in \{(1122);\ (1221);\ (1222);\ (1231);\ (1232);\ (1242);\ (1342);\ (2342)\}$$

Finally, the group L is generated by all one dimensional unipotent elements $\{x_\alpha(r)\}$ such that $\alpha \in \{-(1000);\ -(1100);\ -(1110);\ -(1120)\}$.

We perform 4 root exchange as explained in subsection 2.2.2. First, we exchange $-(1110)$ with (1220), then $-(1120)$ with (1121), $-(1100)$ with (1111) and $-(1000)$ with (1110). Then we expand the integral we obtain along the unipotent group $\{x_{1000}(m_1)x_{1100}(m_2)x_{1120}(m_3)\}$. Thus, integral (106) is equal to

$$\int\limits_{L(\mathbf{A})} \sum_{a,b,c\in F} \int\limits_{U_3(F)\backslash U_3(\mathbf{A})} \int\limits_{U_1(F)\backslash U_1(\mathbf{A})} E(u_3u_2lw_0)\psi_{U_1}(u_1)\psi_{a,b,c}(u_3)du_1du_3dl \tag{107}$$

Here $U_3 = U_{\alpha_2,\alpha_3,\alpha_4}$. The character $\psi_{a,b,c}(u_3)$ is defined as follows. Write an element $u_3 = x_{1000}(m_1)x_{1100}(m_2)x_{1120}(m_3)u_3'$. Then $\psi_{a,b,c}(u_3) = \psi(am_1 + bm_2 + cm_3)$, where $a, b, c \in F$.

There are several cases to consider. First assume that $a = b = c = 0$. Then, in integral (107), the integration over U_3 is the constant term of the function E along the unipotent group U_3. If $a = b = 0$ and $c \neq 0$, then the combined integration over U_1 and U_3 contains as inner integration the Fourier coefficient corresponding to the unipotent orbit $F_4(a_1)$. Finally, if $(a, b) \neq (0, 0)$ then we obtain, as inner integration, the Fourier coefficient corresponding to the unipotent orbit $F_4(a_2)$.

We summarize

Proposition 25. *Let $\mathcal{E}$ denote an automorphic representation of $F_4(\mathbf{A})$ such that:*

1) *The representation $\mathcal{E}$ has no nonzero Fourier coefficients associated with the unipotent*

orbits $F_4(a_1)$ and $F_4(a_2)$ as given above. Also, the representation $\mathcal{E}$ does not support the constant term along the group $U_{\alpha_2,\alpha_3,\alpha_4}$

2) *The representation $\mathcal{E}$ has a nonzero Fourier coefficient associated with the unipotent orbit $F_4(a_3)$ which is given by integral* (105).

Then the representation σ is a nonzero cuspidal representation of the group $\widetilde{SL}_2(\mathbf{A})$.

4.3.4. **The Unipotent Orbit B_2.** The diagram of this unipotent orbit contains nodes which are labeled with a one. Thus the descent in this case is given in terms of the integral (74). Here the theta representation is defined on the group Sp_4. Also the stabilizer is a group of type $A_1 \times A_1$. Since the embedding of $SL_2 \times SL_2$ in Sp_4 does not split, the representation σ is an automorphic representation of $\widetilde{SL}_2(\mathbf{A}) \times \widetilde{SL}_2(\mathbf{A})$.

In the notations of equation (74), let $U_\Delta = U_{\alpha_2,\alpha_3}$. To determine the conditions for the non vanishing and for the cuspidality, we may instead consider integral (76). Thus, in the notations of that integral, let U'_Δ denote the subgroup of U_{α_2,α_3} where we omit the roots (0001) and (0011). The character ψ_{U,u_Δ} is defined as follows. For $u' = x_{1110}(r_1)x_{0122}(r_2)u'_1$ let $\psi_{U,u_\Delta}(u') = \psi(r_1 + r_2)$. With this choice of character the maximal unipotent subgroup of the stabilizer, which is $SL_2 \times SL_2$, is given by $\{x_{0100}(m_1)x_{0120}(m_2)\}$. Denote $V = U_\Delta$ and $V' = U'_\Delta$. We have $h_{B_2}(t) = h(t^6, t^{10}, t^7, t^4)$. The corresponding torus element of the above copy of $SL_2 \times SL_2$ is $h(1, t^2, t, 1)$. Hence the product is $h_{F_4(a_3)}(t)$. Therefore, to study the nonvanishing of this construction we consider the integral

$$\int\limits_{(F\backslash \mathbf{A})^2} \int\limits_{V'(F)\backslash V'(\mathbf{A})} E(v'x_{0100}(m_1)x_{0120}(m_2))\psi_{U,u_\Delta}(v')\psi(am_1 + m_2)dv'dm_1dm_2$$

Here $a \in F^*$. Exchanging the root (1000) by (0110) as explained in subsection 2.2.2 we obtain the integral

$$\int\limits_{\mathbf{A}} \int\limits_{U(F)\backslash U(\mathbf{A})} E(ux_{1000}(r))\psi_{U,a}(u)dudr \tag{108}$$

Here $U = U_{\alpha_1,\alpha_3,\alpha_4}$ and the character $\psi_{U,a}$ is defined as follows. For an element $u = x_{0100}(r_1)x_{1110}(r_2)x_{0120}(r_3)x_{0122}(r_4)u'$ define $\psi_{U,a}(u) = \psi(ar_1 + r_2 + r_3 + r_4)$. In the notations of subsection 2.2.1 this corresponds to the character $\psi_{U_\Delta,A,B}$ with

$$A = \begin{pmatrix} & & 1 \\ & 1 & \\ \alpha & & \end{pmatrix} \qquad B = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

Solving the equations given in (10), we obtain that the only solution is the trivial solution. Hence, the above character $\psi_{U,a}$ is associated with the unipotent orbit $F_4(a_3)$. In the

notations of equation (85) we proved

$$B_2(\mathcal{E}) \circ (2|2)_a = F_4(a_3)_a$$

Here by (2|2) we denote the unipotent orbit of $SL_2 \times SL_2$ which corresponds to the Whittaker coefficient of this group.

To study the cuspidality of the lift, we need to consider two constant terms. One along the unipotent group $\{x_{0120}(r_1)\}$ and the other along $\{x_{0100}(r_2)\}$. However, these two matrices are conjugate under the Weyl element w_3. Moreover, this Weyl element normalizes the group V' and the character ψ_{U,u_Δ}. Hence it is enough to consider the integral

$$\int\limits_{F\backslash \mathbf{A}} \int\limits_{V'(F)\backslash V'(\mathbf{A})} E(v'x_{0120}(m))\psi_{U,u_\Delta}(v')dv'dm$$

Let $w_0 = w[123421]$ and conjugate in the above integral by this Weyl element. Then the above integral is equal to

$$\int\limits_{L(F)\backslash L(\mathbf{A})} \int\limits_{U_2(F)\backslash U_2(\mathbf{A})} \int\limits_{U_1(F)\backslash U_1(\mathbf{A})} E(u_1u_2lw_0)\psi_{U_1}(u_1)du_1du_2dl$$

Here U_1 is the unipotent subgroup of Sp_6 which is generated by $\{x_\alpha(r)\}$ where

$$\alpha \in \{(0001);\ (0110);\ (0011);\ (0111);\ (0120);\ (0121);\ (0122)\}$$

The character ψ_{U_1} is defined as follows. Write $u_1 = x_{0001}(r_1)x_{0120}(r_2)u_1'$. Then $\psi_{U_1}(u_1) = \psi(r_1 + r_2)$. The group U_2 is generated by all $\{x_\alpha(r)\}$ such that

$$\alpha \in \{(1122);\ (1222);\ (1231);\ (1232);\ (1242);\ (1342);\ (2342)\}$$

Finally, the group L is generated by all $\{x_\alpha(r)\}$ such that

$$\alpha \in \{-(1000);\ -(1100);\ -(1110);\ -(1120);\ -(1220)\}$$

First we exchange roots as follows. Exchange $-(1220)$ with (1221), $-(1120)$ with (1121), $-(1100)$ with (1220), $-(1110)$ with (1111) and $-(1000)$ with (1120). Then we perform a Fourier expansion along the roots (0100); (1100) and (1110). Then, the above integral is equal to

$$\int\limits_{L(\mathbf{A})} \sum_{a,b,c} \int\limits_{U_4(F)\backslash U_4(\mathbf{A})} \int\limits_{U_3(F)\backslash U_3(\mathbf{A})} E(u_4u_3lw_0)\psi_{U_3,a}(u_3)\psi_{U_4,b,c}(u_4)du_4du_3dl \tag{109}$$

Here U_3 is the unipotent subgroup of Sp_6 generated by U_1 and the group $\{x_{0100}(r)\}$. The character $\psi_{U_3,a}$ is defined as follows. Write $u_3 = x_{0001}(r_1)x_{0120}(r_2)x_{0100}(r_3)u_3'$. Then $\psi_{U_3,a}(u_3) = \psi(r_1+r_2+ar_3)$. The group U_4 is generated by U_2 and the unipotent group $\{x_{1100}(r_1)x_{1110}(r_2)\}$. To define the character $\psi_{U_4,b,c}$ write $u_4 = x_{1100}(r_1)x_{1110}(r_2)u_4'$. Then $\psi_{U_4,b,c}(u_4) = \psi(br_1+cr_2)$.

In the notations of subsection 2.2.1 which describes the characters of $F_4(a_2)$, the character corresponding to the integral (109), corresponds to the character $\psi_{U_\Delta,A,\gamma_1,\gamma_2}$ where

$$A = \begin{pmatrix} 1 & a \\ & b \\ c & \\ & \end{pmatrix} \qquad (\gamma_1, \gamma_2) = (1, 0)$$

There are several cases. Assume first that $b, c \neq 0$. Then, we obtain as inner integration, a Fourier coefficient which correspond to the unipotent orbit $F_4(a_2)$. Indeed, from the description of the action on the set of characters, as described in subsection 2.2.1 it follows that the stabilizer is a finite group. The same happens if $a, c \neq 0$ but $b = 0$ and similarly if $a, b \neq 0$ but $c = 0$. In all other cases we either get a Fourier coefficient which corresponds to a unipotent orbit which is either $F_4(a_1)$ or F_4, or we get the constant term along the unipotent radical of the maximal parabolic subgroup whose Levi part is GSp_6.

We proved,

Proposition 26. *Let $\mathcal{E}$ denote an automorphic representation of $F_4(\mathbf{A})$ such that:*
1) *The representation $\mathcal{E}$ has no nonzero Fourier coefficients associated with the unipotent orbits F_4, $F_4(a_1)$ and $F_4(a_2)$ as given above. Also, the representation $\mathcal{E}$ does not support the constant term along the group $U_{\alpha_2,\alpha_3,\alpha_4}$*
2) *The representation $\mathcal{E}$ has a nonzero Fourier coefficient associated with the unipotent orbit $F_4(a_3)$ which is given by integral* (108)*.*

Then the representation σ is a nonzero cuspidal representation of the group $\widetilde{SL_2}(\mathbf{A}) \times \widetilde{SL_2}(\mathbf{A})$.

4.3.5. **The Unipotent Orbit $A_2 + \widetilde{A}_1$.** The diagram corresponding to this orbits has nodes labelled with ones, and hence we use integral (74) with a suitable theta representation. In this case we denote $U_\Delta = U_{\alpha_1,\alpha_2,\alpha_4}$ and let U'_Δ denote the subgroup of U_Δ generated by all roots in U_Δ omitting $\alpha \in \{(0010); (0110); (0011)\}$. Thus dim $U'_\Delta = 17$. We define the character ψ_{U,u_Δ} as follows. Write $u' \in U'_\Delta$ as $u' = x_{0122}(r_1)x_{1121}(r_2)x_{1220}(r_3)u''$. Then define $\psi_{U,u_\Delta}(u') = \psi(r_1 + r_2 + r_3)$. The stabilizer of this orbit is the group SL_2, and we can choose the embedding inside F_4, such that its standard unipotent subgroup is the group $x(r) = x_{1000}(r)x_{0100}(\eta_1 r)x_{1100}(\eta_2 r)x_{0001}(r)$. Here $\eta_i \in F^*$.

The torus corresponding to this orbit is given by $h_{A_2+\widetilde{A}_1}(t) = h(t^4, t^8, t^6, t^3)$. The torus of the above SL_2 is given by $h(t^2, t^2, 1, t)$, and hence their product is $h(t^6, t^{10}, t^6, t^4)$. Conjugating by the Weyl element $w[23]$, we obtain the torus attached to the orbit $F_4(a_3)$.

Given, $z \in F$, we consider the integral

$$\int_{F\backslash \mathbf{A}} \int_{V'(F)\backslash V'(\mathbf{A})} E(v'x(r))\psi_{U,u_\Delta}(v')\psi(zr)dv'dr \tag{110}$$

where we denoted $V' = U'_\Delta$. Since the computations in this example are quite involved, we will only sketch part of them. In other words, we will show that when $z \neq 0$ then we do obtain the Fourier coefficient associated with the unipotent orbit $F_4(a_3)$. However, we also get other terms which corresponds to unipotent orbits which are greater than $F_4(a_3)$, and also some constant terms.

We start with the root exchange, (0120) with (1100), then (0121) with (0001) and then (1120) with (0100). Conjugating by $w[23]$ we obtain the integral

$$\int_{L(\mathbf{A})} \int_{U_1(F)\backslash U_1(\mathbf{A})} E(u_1 w[23]l)\psi_{U_1,z}(u_1)du_1 dl \tag{111}$$

Here L is the unipotent group generated by all $\{x_\alpha(r)\}$ where $\alpha \in \{(0120);\ (0121);\ (1120)\}$. The group U_1 is the subgroup of $U_{\alpha_1,\alpha_3,\alpha_4}$ omitting the two roots (0100) and (0110). Thus $\dim U_1 = 18$. To define the character $\psi_{U_1,z}$, write $u_1 = x_{1100}(r_1)x_{1111}(r_2 x_{1120}(r_3)x_{0122}(r_4)u'_1$. Then $\psi_{U_1,z}(u_1) = \psi(zr_1 + r_2 + r_3 + r_4)$.

Next we expand the above integral along $\{x_{0100}(l_1)x_{0110}(l_2)\}$, and we obtain that integral (111) is zero for all choice of data if and only if the integral

$$\sum_{m,n\in F} \int_{U(F)\backslash U(\mathbf{A})} E(u)\psi_{U,z,m,n}(u)du$$

is zero for all choice of data. Here, $U = U_{\alpha_1,\alpha_3,\alpha_4}$, and the character $\psi_{U,z,m,n}$ is defined as follows. Write

$$u = x_{1100}(r_1)x_{1111}(r_2)x_{1120}(r_3)x_{0122}(r_4)x_{0100}(r_5)x_{0110}(r_6)u'_1$$

Then $\psi_{U,z,m,n} = \psi(zr_1 + r_2 + r_3 + r_4 + mr_5 + nr_6)$. In general position, this Fourier coefficient corresponds to the unipotent orbit $F_4(a_3)$. Indeed, in the notations of subsection 2.2.1 the above character corresponds to the character $\psi_{U_\Delta,A,B}$ with

$$A = \begin{pmatrix} & m & n \\ & & m \\ 1 & & \end{pmatrix} \qquad B = \begin{pmatrix} 1 & & z \\ & 1 & \\ & & 1 \end{pmatrix}$$

Solving the equations (10) with

$$g_1 = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \qquad h_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we obtain four variables b_2, c_1, c_2, c_3 which satisfy the system

$$zb_2 - (2n+z^2)c_1 - 3mc_2 - zc_3 = 0$$

$$3mb_2 - mzc_1 - nc_2 - 3mc_3 = 0$$

$$2nb_2 + (3m^2 - nz)c_1 + 2mzc_2 - 2nc_3 = 0$$

All other variables which appear in (g_1, h_1) are determined by these four variables. Since we always have the trivial solution, we set $c_3 = b_2 + c_4$, where c_4 is a new variable. We then obtain the system

$$-(2n+z^2)c_1 - 3mc_2 - zc_4 = 0$$

$$-mzc_1 - nc_2 - 3mc_4 = 0$$

$$(3m^2 - nz)c_1 + 2mzc_2 - 2nc_4 = 0$$

This system has a nontrivial solution if and only if the determinant of the matrix corresponding to this system is zero. In this case we obtain the determinant

$$f(m,n,z) = -27m^4 + 18nm^2z + 4m^2z^3 + 4n^3 + n^2z^2$$

Thus, for those values of m, n and z such that $f(m,n,z) \neq 0$, the above Fourier coefficient corresponds to the unipotent orbit $F_4(a_3)$. To analyze the other orbits we need to solve the equation $f(m,n,z) = 0$. We claim that in this case we obtain Fourier coefficients which are associated to all unipotent orbits which are greater than $F_4(a_3)$. We demonstrate this claim in the case when $m = n = 0$. In other words, we consider the Fourier coefficient

$$\int\limits_{U(F)\backslash U(\mathbf{A})} E(u)\psi_{U,z,0,0}(u)du \tag{112}$$

For fixed $s_1, s_2 \in F$, this integral is zero for all choice of data if and only if the integral

$$\int\limits_{U(F)\backslash U(\mathbf{A})} E(ux_{-0011}(s_1)x_{-1000}(s_2))\psi_{U,z,0,0}(u)du$$

is zero for all choice of data. Conjugate these two elements to the left. Recall that $U = U_{\alpha_1,\alpha_3,\alpha_4}$. Since these elements are inside the Levi subgroup of $P_{\alpha_1,\alpha_3,\alpha_4}$, this conjugation preserves the group U. We do however need to determine how this conjugation effects the character $\psi_{U,z,0,0}$. To do that we consider the conjugation

$$x_{-0011}(-s_1)x_{-1000}(-s_2)x_{1100}(r_1)x_{1111}(r_2)x_{0122}(r_4)x_{1122}(m)x_{-0011}(s_1)x_{-1000}(s_2)$$

Conjugate $x_{-0011}(s_1)$ across $x_{1122}(m)$. We obtain

$$x_{-0011}(-s_1)x_{-1000}(-s_2)x_{1100}(r_1 - ms_1^2)x_{1111}(r_2 + ms_1)x_{0122}(r_4)x_{-0011}(s_1)x_{1122}(m)x_{-1000}(s_2)u_1$$

Here $u_1 \in U$ is a product of one dimensional unipotent subgroups $\{x_\alpha(r)\}$ such that $\psi_{U,z,0,0}(u_1) = 1$ and α is not any of the above roots. Changing variables $r_1 \to r_1 + ms_1^2$ and then $r_2 \to r_2 - ms_1$, we obtain the character

$$\psi_{U,z,s_1,s_2}(u) = \psi(zr_1 + r_2 + r_3 + r_4 + zms_1^2 - ms_1)$$

We further conjugate $x_{-0011}(s_1)$ to the left, and we obtain

$$x_{-1000}(-s_2)x_{1100}(r_1 + 2r_2s_1)x_{1111}(r_2)x_{0122}(r_4)x_{1122}(m)x_{-1000}(s_2)u_2$$

where u_2 is defined in a similar way as u_1. Changing variables $r_1 \to r_1 - 2r_2s_1$ we obtain the character

$$\psi'_{U,z,s_1,s_2}(u) = \psi(zr_1 + r_2(1 - 2zs_1) + r_3 + r_4 + m(zs_1^2 - s_1))$$

Finally, conjugating by $x_{-1000}(s_2)$ we obtain

$$x_{1100}(r_1)x_{1111}(r_2)x_{0122}(r_4 - ms_2)x_{1122}(m)u_3$$

Changing variables $r_4 \to r_4 + ms_2$ we obtain the character

$$\psi''_{U,z,s_1,s_2}(u) = \psi(zr_1 + r_2(1 - 2zs_1) + r_3 + r_4 + m(zs_1^2 - s_1 + s_2))$$

Choosing s_1 and s_2 such that $1 - 2zs_1 = 0$ and $zs_1^2 - s_1 + s_2 = 0$, we deduce that integral (112) is zero for all choice of data if and only if the integral

$$\int\limits_{U(F)\backslash U(\mathbf{A})} E(u)\psi_{U,z}(u)du$$

is zero for all choice of data. Here $\psi_{U,z}$ is defined as follows. For $u \in U$, write $u = x_{1100}(r_1)x_{1120}(r_2)x_{0122}(r_3)u'$. Then $\psi_{U,z}(u) = \psi(zr_1 + r_2 + r_3)$. To proceed, we conjugate by the Weyl element $w_0 = w[432341]$. Thus, the above integral is equal to

$$\int\limits_{L(F)\backslash L(\mathbf{A})} \int\limits_{U_2(F)\backslash U_2(\mathbf{A})} \int\limits_{U_1(F)\backslash U_1(\mathbf{A})} E(u_2u_1lw_0)\psi_{U_1,z}(u_1)du_1du_2dl$$

Here U_1 is the unipotent subgroup of $GSpin_7$ generated by all $\{x_\alpha(r)\}$ where

$$\alpha \in \{(0100);\ (0110);\ (0120);\ (1000);\ (1100);\ (1110);\ (1120);\ (1220)\}$$

The character $\psi_{U_1,z}$ is defined as follows. Write $u_1 = x_{0100}(r_1)x_{0120}(r_1)x_{1000}(r_3)u_1'$. Then $\psi_{U_1,z} = \psi(zr_1 + r_2 + r_3)$. The group U_2 is generated by all $\{x_\alpha(r)\}$ such that

$$\alpha \in \{(1111);\ (1121);\ (1221);\ (1231);\ (1222);\ (1232);\ (1242);\ (1342);\ (2341)\}$$

and the group L is generated by $\{x_\alpha(r)\}$ where $\alpha \in \{-(0011);\ -(0001);\ -(0122)\}$.

As explained in subsection 2.2.2 we exchange the root $-(0122)$ with (1122), then $-(0001)$ with (0121) and $-(0011)$ with (0111). Thus, the above integral is equal to

$$\int\limits_{L(\mathbf{A})}\int\limits_{U_3(F)\backslash U_3(\mathbf{A})}\int\limits_{U_1(F)\backslash U_1(\mathbf{A})} E(u_2u_1lw_0)\psi_{U_1,z}(u_1)du_1du_2dl$$

Here U_3 is the unipotent group generated by U_2 and $\{x_\alpha(r)\}$ where α is a root in the set $\{(0111);\ (0121);\ (1122)\}$. The next step is to expand the above integral along $\{x_{1122}(r)\}$. We obtain

$$\int\limits_{L(\mathbf{A})}\sum_{a\in F}\int\limits_{F\backslash\mathbf{A}}\int\limits_{U_3(F)\backslash U_3(\mathbf{A})}\int\limits_{U_1(F)\backslash U_1(\mathbf{A})} E(u_2x_{0122}(m)u_1lw_0)\psi_{U_1,z}(u_1)\psi(am)du_1dmdu_2dl$$

If $a\neq 0$ then the inner integration is a Fourier coefficient which corresponds to the unipotent orbit $F_4(a_2)$. When $a=0$ we further expand along the unipotent group $\{x_{0001}(m_1)x_{0011}(m_2)\}$. Any nontrivial character corresponding to this expansion yields a Fourier coefficient attached to the unipotent orbit $F_4(a_1)$. The trivial character contributes the integral

$$\int\limits_{L(\mathbf{A})}\int\limits_{U_1(F)\backslash U_1(\mathbf{A})} E^{U_{\alpha_1,\alpha_2,\alpha_3}}(u_1lw_0)\psi_{U_1,z}(u_1)du_1dl \tag{113}$$

To summarize this case, we deduce that the Fourier coefficient given by integral (110), when expressed in terms of Fourier coefficients associated with unipotent orbits of F_4, has a contribution from all unipotent orbits which are greater than the orbit $F_4(a_3)$. We also get the constant term (113) as a summand.

4.3.6. **The Unipotent Orbit $\widetilde{A}_2$.** Let $V=U_{\alpha_1,\alpha_2,\alpha_3}$. Thus $\dim V=15$. We define a character ψ_V as follows. Write $v=x_{0121}(r_1)x_{1111}(r_2)v'$. Then define $\psi_V(v)+\psi(r_1+r_2)$. As follows from [C], the stabilizer inside $Spin_7$ of this character is the exceptional group G_2. The embedding of the standard unipotent subgroup of G_2 is given as follows

$$\{x_{1000}(m)x_{0010}(-m);\ x_{0100}(m);\ x_{1100}(m)x_{0110}(-m);\ x_{1110}(m)x_{0120}(-m);\ x_{1120}(m);\ x_{1220}(m)\}$$

The unipotent subgroups which corresponds to the simple roots are $\{x_{1000}(m)x_{0010}(-m)\}$ and $\{x_{0100}(m)\}$. The group G_2 has two maximal parabolic subgroups, and we will denote by U_1 and by U_2 their unipotent radicals. More precisely, we let

$$U_1=\{x_{0100}(m);\ x_{1100}(m)x_{0110}(-m);\ x_{1110}(m)x_{0120}(-m);\ x_{1120}(m);\ x_{1220}(m)\}$$

and

$$U_2=\{x_{1000}(m)x_{0010}(-m);\ x_{1100}(m)x_{0110}(-m);\ x_{1110}(m)x_{0120}(-m);\ x_{1120}(m);\ x_{1220}(m)\}$$

We start by computing the unipotent radical along U_1. We expand the constant term along $\{x_{1000}(m)x_{0010}(-m)\}$, and we obtain the integral

$$\int\limits_{U_1(F)\backslash U_1(\mathbf{A})} \sum_{\gamma\in F} \int\limits_{F\backslash\mathbf{A}} \int\limits_{V(F)\backslash V(\mathbf{A})} E(vx_{1000}(m)x_{0010}(-m)u_1)\psi_V(v)\psi(\gamma m)dmdu_1dv \tag{114}$$

Write integral (114) as

$$\int\limits_{(F\backslash\mathbf{A})^5} \sum_{\gamma\in F} \int\limits_{F\backslash\mathbf{A}} \int\limits_{V(F)\backslash V(\mathbf{A})} E(vz(m_1,m_2,m_3)y(l_1,l_2,l_3))\psi_V(v)\psi(\gamma m_1)dm_idl_jdv \tag{115}$$

Here $z(m_1,m_2,m_3) = x_{1000}(m_1)x_{0010}(-m_1)x_{1100}(m_2)x_{0110}(-m_2)x_{1110}(m_3)x_{0120}(-m_3)$, and $y(l_1,l_2,l_3) = x_{0100}(l_1)x_{1120}(l_2)x_{1220}(l_3)$. Next we consider a certain Fourier expansion, and we apply the root exchange process as explained in subsection 2.2.2.

We start by expanding the above integral along the unipotent group $\{x_{1110}(r_3)\}$. We then apply the root exchange process with the unipotent group $\{x_{0111}(p_3)\}$. Thus, in the notions introduced right after (26), we exchange the root (1110) by the root (0111). We repeat this process two more times. First we exchange (1100) by (0011), and then (1000) by (0001). After that, we conjugate by the Weyl element $w_0 = w[13234]$. Then integral (115) is zero for all choice of data if and only if for each $\gamma \in F$, the integral

$$\int\limits_{U_3(F)\backslash U_3(\mathbf{A})} \int\limits_{V_3(F)\backslash V_3(\mathbf{A})} E(v_3u_3)\psi_{V_{3,\gamma}}(v_3)dv_3du_3$$

is zero for all choice of data. Here, $\gamma \in F$, and V_3 is the unipotent subgroup generated by $\{x_\alpha(r)\}$, where α is in the set of roots

$$\{(0100);\ (0001);\ (0011);\ (0110);\ (0120);\ (0111);\ (0121);\ (0122)\}$$

Thus V_3 is a subgroup of Sp_6 embedded inF_4 as the Levi part of $P_{\alpha_2,\alpha_3,\alpha_4}$. Denote $U(C_3) = U_{\alpha_2,\alpha_3,\alpha_4}$. The group U_3 is the subgroup of $U(C_3)$ generated by all roots in $U(C_3)$ accept for the roots (1120) and (1000). Thus $\dim U_3 = 13$. The character $\psi_{V_{3,\gamma}}$ is defined as follows. Write $v_3 = x_{0001}(r_1)x_{0110}(r_2)x_{0120}(r_3)v_3'$. Then $\psi_{V_{3,\gamma}}(v_3) = \psi(r_1 + r_2 + \gamma r_3)$. Next we expand along the unipotent group $\{x_{1120}(r)\}$. Thus, we obtain the integral

$$\sum_{\beta\in F} \int\limits_{U_3(F)\backslash U_3(\mathbf{A})} \int\limits_{F\backslash\mathbf{A}} \int\limits_{V_3(F)\backslash V_3(\mathbf{A})} E(x_{1120}(r)v_3u_3)\psi_{V_{3,\gamma}}(v_3)\psi(\beta r)dv_3drdu_3 \tag{116}$$

There are two cases. First, the contribution of each summand when $\beta \neq 0$ to the integral (116), produces a Fourier coefficient which corresponds to the unipotent orbit $F_4(a_2)$. In the summand, where $\beta = 0$, we further expand along $\{x_{1000}(r)\}$. Depending on γ, the nontrivial orbit contributes Fourier coefficients which corresponds to unipotent orbits $F_4(a_1)$ and F_4.

The trivial orbit produces an integral of the type

$$\int\limits_{V_3(F)\backslash V_3(\mathbf{A})} E^{U(C_3)}(v_3)\psi_{V_3,\gamma}(v_3)dv_3$$

The computation of the constant term along the unipotent group U_2 is similar and gives the same result. We record this as

Proposition 27. *Suppose that the representation $\mathcal{E}$ has no nonzero Fourier coefficients which corresponds to the unipotent orbits F_4, $F_4(a_1)$ and $F_4(a_2)$. Suppose also that $E^{U(C_3)}$ is zero for all functions $E \in \mathcal{E}$. Then the automorphic representation σ is a cuspidal representation.*

Next we consider the nonvanishing of the descent. Here we have two cases to consider. The first, is when the lift is generic. The integral we consider is

$$\int\limits_{(F\backslash\mathbf{A})^6}\int\limits_{V(F)\backslash V(\mathbf{A})} E(vz(m_1,m_2,m_3)y(l_1,l_2,l_3))\psi_V(v)\psi(l_1+m_1)dm_i dl_j dv$$

where the notations are defined in (115). As in the part of the cuspidality, we start with some roots exchange (See subsection 2.2.2). First, we exchange (0001) by (1110), then (0011) by (1100) and (0111) by (1000). Thus, the above integral is equal to

$$\int\limits_{\mathbf{A}^3}\int\limits_{Y(F)\backslash Y(\mathbf{A})}\int\limits_{V_1(F)\backslash V_1(\mathbf{A})} E(v_1yl(r_1,r_2,r_3))\psi_{V_1}(v_1)\psi_Y(y)dydv_1dr_k \tag{117}$$

Here V_1 is the subgroup of V consisting of all roots in V omitting the roots (0001); (0011) and (0111). Thus $\dim V_1 = 12$. Next, Y is the maximal unipotent subgroup of $Spin_7$ as embedded in F_4 as the Levi part of $P_{\alpha_1,\alpha_2,\alpha_3}$. Thus, the roots in Y are all nine roots in F_4 of the form $n_1\alpha_1+n_2\alpha_2+n_3\alpha_3$. The character ψ_Y is defined as $\psi_Y(y) = \psi_Y(x_{1000}(r_1)x_{0100}(r_2)y') = \psi(r_1+r_2)$. Finally, we have $l(r_1,r_2,r_3) = x_{0001}(r_1)x_{0011}(r_1)x_{0111}(r_3)$.

We have $h_{\widetilde{A}_2}(t) = h(t^4,t^8,t^6,t^4)$. We are computing the Whittaker coefficient of the lift, which corresponds to the unipotent orbit of G_2 whose label is G_2. The corresponding torus, as embedded in F_4, is $h(t^6,t^{10},t^6,1)$. Thus the product of these two tori is $h(t^{10},t^{18},t^{12},t^4)$. Conjugating by $w[234]$ we get $h(t^{10},t^{20},t^{14},t^8)$ which is equal to $h_{F_4(a_2)}(t)$. It is convenient to conjugate by $w[3234]$ and thus, integral (117) is equal to

$$\int\limits_{\mathbf{A}^3}\int\limits_{(F\backslash\mathbf{A})^2}\int\limits_{V_2(F)\backslash V_2(\mathbf{A})} E(v_2x_{1000}(m_1)x_{-0120}(m_2)w[3234]l(r_1,r_2,r_3))\psi_{V_2}(v_2)dv_2dm_idr_k$$

Here, V_2 is the unipotent subgroup of F_4 whose dimension is 19 and consists of all positive roots in F_4 omitting the roots (1000); (0010); (0110); (0120) and (0121). The character ψ_{V_2} is defined by $\psi_{V_2}(v_2) = \psi_{V_2}(x_{0001}(r_1)x_{0100}(r_2)x_{1110}(r_3)x_{1120}(r_4)v_2') = \psi(r_1+r_2+r_3+r_4)$.

Next we exchange the root $-(0120)$ by (0121) and (0110) by (1000). Then we expand the integral along the unipotent subgroup $\{x_{0120}(r)\}$. Thus, the above integral is equal to

$$\sum_{\beta\in F}\int_{\mathbf{A}^5}\int_{V_3(F)\backslash V_3(\mathbf{A})} E(v_3x_{1000}(m_1)x_{-0120}(m_2)w[3234]l(r_1,r_2,r_3))\psi_{V_3,\beta}(v_3)dv_3dm_idr_k$$

Here V_3 is the unipotent subgroup of F_4 which consists of all positive roots omitting the two roots (1000) and (0010). Thus, dim $V_3 = 22$. Also

$$\psi_{V_3,\beta}(v_3) = \psi_{V_3,\beta}(x_{0001}(r_1)x_{0100}(r_2)x_{1110}(r_3)x_{0120}(r_4)x_{1120}(r_5)v_3') = \psi(r_1 + r_2 + r_3 + \beta r_4 + r_5)$$

Arguing as in [Ga-S], the above integral is nonzero for some choice of data if and only if the integral

$$\sum_{\beta\in F}\int_{V_3(F)\backslash V_3(\mathbf{A})} E(v_3)\psi_{V_3,\beta}(v_3)dv_3 \tag{118}$$

is not zero for some choice of data. In the notations of subsection 2.2.1 the group $V_3 = U_{\alpha_1,\alpha_3}$, and the character $\psi_{V_3,\beta}$ is defined by

$$\psi_{V_3,\beta}(v_3) = \psi(z(m_1, m_2)y(r_1, \ldots, r_6)v_3') = \psi(m_1 + r_1 + r_4 + \beta r_5 + r_6)$$

For $\gamma \in F$, write $E(v_3) = E(v_3x_{0010}(\gamma)x_{0010}(-\gamma))$ and conjugate the element $x_{0010}(\gamma)$ to the left across v_3. Changing variables will change the character $\psi_{V_3,\beta}$. We write down the commutation relations needed for the above conjugation $[x_{1110}(r), x_{0010}(s)] = x_{1120}(2rs)$; $[x_{0110}(r), x_{0010}(s)] = x_{0120}(2rs)$; $[x_{1100}(r), x_{0010}(s)] = x_{1110}(rs)x_{1120}(rs^2)$ and the relation $[x_{0100}(r), x_{0010}(s)] = x_{0110}(rs)x_{0120}(rs^2)$. The conjugation $x_{0010}(-\gamma)v_3x_{0010}(\gamma)$ transforms the character $\psi_{V_3,\beta}$ to the character

$$\psi(m_1 + (1 + \beta\gamma^2)r_1 + (\gamma^2 - \gamma)r_2 - 2\beta\gamma r_3 + (1 - 2\gamma)r_4 + \beta r_5 + r_6)$$

Notice that only when $\gamma = 1$ and $\beta = -1$, then the coefficients of r_1 and r_2 are zero. Choose $\gamma = 1$. We separate the sum in (118) into two summands. First, consider the contribution when $\beta = -1$. Performing the above conjugation, we obtain

$$\int_{V_3(F)\backslash V_3(\mathbf{A})} E(v_3x_{0010}(1))\psi_1(v_3)dv_3$$

where

$$\psi_1(v_3) = \psi(m_1 + 2r_3 - r_4 - r_5 + r_6) = \psi(m_1 + \text{tr } \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} r_3 & r_4 \\ r_5 & r_6 \end{pmatrix})$$

The group $GL_2(F)$ which contains the group $SL_2(F) =< x_{\pm 1000}(r) >$ acts on the group V_3. Since the matrix $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ is invertible, we can find a suitable matrix in $\delta \in GL_2(F)$, such

that the above integral is equal to

$$\int\limits_{V_3(F)\backslash V_3(\mathbf{A})} E(v_3\delta x_{0010}(1))\psi_2(v_3)dv_3$$

Here $\psi_2(v_3) = \psi(m_1 + r_4 + r_5)$.

Consider the Weyl element $w_0 = w[1234213]$. Using the fact that $E(g) = E(w_0g)$, we conjugate this Weyl element to the right in the above integral, and we obtain

$$\int\limits_{(F\backslash\mathbf{A})^5}\int\limits_{L(F)\backslash L(\mathbf{A})}\int\limits_{V_4(F)\backslash V_4(\mathbf{A})} E(v_4la(m_1,\ldots,m_5)\mu)\psi_L(l)dv_4dldm_i$$

Here $\mu = w_0\delta x_{0010}(1)$, and L is the maximal unipotent subgroup of Sp_6 embedded in F_4 as the Levi part of $P_{\alpha_2,\alpha_3,\alpha_4}$. The character ψ_L is the Whittaker character of L. In other words,

$$\psi_L(l) = \psi_L(x_{0100}(l_1)x_{0010}(l_2)x_{0001}(l_3)l') = \psi(l_1 + l_2 + l_3)$$

The group V_4 is the unipotent group generated by all $\{x_\alpha(r)\}$ where α is a root in

$$\{(1122);\ (1221);\ (1222);\ (1231);\ (1232);\ (1241);\ (1342);\ (2342)\}$$

Finally, we have

$$a(m_1,\ldots,m_5) = x_{-1000}(m_1)x_{-1100}(m_2)x_{-1110}(m_3)x_{-1111}(m_4)x_{-1120}(m_5)$$

Next we consider five root exchanges. First, we exchange -1120 by 1220. Then, -1111 by 1121, -1110 by 1120, -1100 by 1110 and -1000 by 1100. After these roots exchange, we expand the integral along $\{x_{1000}(r_1)x_{1111}(r_2)\}$. Thus, the above integral is equal to

$$\int\limits_{\mathbf{A}^5}\sum_{\beta,\gamma\in F}\int\limits_{L(F)\backslash L(\mathbf{A})}\int\limits_{V_5(F)\backslash V_5(\mathbf{A})} E(v_5la(m_1,\ldots,m_5)\mu)\psi_L(l)\psi_{V_5,\beta,\gamma}(v_5)dv_5dldm_i$$

Here $V_5 = U(C_3)$ where $U(C_3)$ was defined right before equation (116). Also, we define the character $\psi_{V_5,\beta,\gamma}(v_5) = \psi_L(x_{1000}(r_1)x_{1111}(r_2)v_5') = \psi(\beta r_1 + \gamma r_2)$. There are several cases to consider. First, if $(\beta,\gamma) = (0,0)$ then we obtain the integral

$$\int\limits_{\mathbf{A}^5}\int\limits_{L(F)\backslash L(\mathbf{A})} E^{U(C_3)}(v_5la(m_1,\ldots,m_5)\mu)\psi_L(l)dldm_i \tag{119}$$

When $\gamma \neq 0$, then after conjugating by the Weyl element $w[21]$ we obtain a Fourier coefficient corresponding to the unipotent orbit $F_4(a_1)$. When $\gamma = 0$ and $\beta \neq 0$, we obtain a Fourier coefficient which corresponds to the unipotent orbit F_4.

Returning to integral (118), so far we analyzed the contribution from the term $\beta = -1$. We still need to consider the integral

$$\sum_{-1\neq\beta\in F}\int\limits_{V_3(F)\backslash V_3(\mathbf{A})} E(v_3)\psi_{V_3,\beta}(v_3)dv_3$$

It follows from the description of the action on the group characters of $V_3(F)\backslash V_3(\mathbf{A})$, as given in subsection 2.2.1, that each summand in the above integral is a Fourier coefficient associated with the unipotent orbit $F_4(a_2)$. This completes the computations of the Whittaker coefficient of the descent.

The next case to consider is when the descent has no Whittaker coefficient. In other words, the Fourier coefficient corresponding to the unipotent orbit whose label is G_2, is zero for all choice of data. In this case, since σ is a cuspidal representation, it has a nonzero Fourier coefficient associated with the unipotent orbit $G_2(a_1)$. These Fourier coefficients are described in [J-R]. Consider the unipotent group U_1 introduced at the beginning of this subsection. We introduce coordinates on this group as follows. Let

$$m(r_1,\ldots,r_5)=x_{0100}(r_1)x_{1100}(-r_2)x_{0110}(r_2)x_{1110}(-r_3)x_{0120}(r_3)x_{1120}(r_4)x_{1220}(r_5)$$

Following [J-R], we defined three characters on this group. For $u\in U$ define $\psi_1(u)=\psi(r_2+r_3)$; $\psi_{2,a}(u)=\psi(ar_1+r_3)$ and $\psi_{3,b,c}(u)=\psi(cr_1+br_2+r_4)$. Here $a,b,c\in F^*$.

As above, the one dimensional torus corresponding to the unipotent orbit $\widetilde{A}_2$ is $h_{\widetilde{A}_2}(t)=h(t^4,t^8,t^6,t^4)$ and $h_{G_2(a_1)}(t)=h(t^2,t^4,t^2,1)$. Hence the product of these two tori elements is $h_{F_4(a_3)}(t)=h(t^6,t^{12},t^8,t^4)$. The Fourier coefficient we need to calculate is given by

$$\int\limits_{U_1(F)\backslash U_1(\mathbf{A})}\ \int\limits_{V(F)\backslash V(\mathbf{A})} E(vu_1)\psi_V(v)\psi_{U_1}(u_1)dvdu_1$$

where ψ_{U_1} is any one of the three type of characters introduced above. As in the above computations, we first perform two root exchange as explained in subsection 2.2.2. First, we exchange the root (0001) with the root (1110), and then exchange the root (0011) with the root (1100). Thus, the above integral is equal to

$$\int\limits_{\mathbf{A}^2}\ \int\limits_{U_\Delta(F)\backslash U_\Delta(\mathbf{A})} E(ux_{0001}(r_1)x_{0011}(r_2))\psi_{U_\Delta}(u)dr_idu$$

Here $\Delta=\{\alpha_1,\alpha_2,\alpha_4\}$ and ψ_{U_Δ} is a character of $U_\Delta(F)\backslash U_\Delta(\mathbf{A})$ which is determined by the character ψ_{U_1} as follows. Write an element $u\in U_\Delta$ as $u=y(r_1,\ldots,r_6)z(m_1,\ldots,m_6)u'$ as right before (8) in subsection 2.2.1. If $\psi_{U_1}=\psi_1$, then

$$\psi_{U_\Delta}(u)=\psi_{U_\Delta}(y(r_1,\ldots,r_6)z(m_1,\ldots,m_6)u')=\psi(r_5+m_1+m_2+m_4)$$

If $\psi_{U_1}=\psi_{2,a}$ then $\psi_{U_\Delta}(u)=\psi(ar_1+r_5+m_2+m_4)$. Finally, if $\psi_{U_1}=\psi_{3,b,c}$ then $\psi_{U_\Delta}(u)=\psi(cr_1+r_5+bm_1+m_3+m_4)$.

We summarize

Proposition 28. *Let $\mathcal{E}$ denote an automorphic representation of $F_4(\mathbf{A})$, and consider its descent to the exceptional group $G_2(\mathbf{A})$.*

a) *Then, the Whittaker coefficient of the descent is a sum of Fourier coefficients corresponding to the unipotent orbits* $F_4(a_2)$, $F_4(a_1)$, F_4 *and the constant term integral* (119). *In other words we have*

$$\widetilde{A}_2(\mathcal{E})\circ G_2 = F_4(a_2) + F_4(a_1) + F_4 + \mathcal{CT}_{F_4,P_{\alpha_2,\alpha_3,\alpha_4}}[(6)_{Sp_6}]$$

b) *The Fourier coefficient of the decent which corresponds to the Fourier coefficient of* G_2 *whose label is* $G_2(a_1)$ *corresponds to the Fourier coefficient* $F_4(a_3)$. *In other words*

$$\widetilde{A}_2(\mathcal{E})\circ G_2(a_1) = F_4(a_3)$$

5. Construction of Small Representations in F_4

In this Section we construct a few examples of small representations $\mathcal{E}$ defined on the group $F_4(\mathbf{A})$. By definition, we define a representation to be a small representation if it is not generic. We will consider two examples which are constructed by means of residue representations of Eisenstein series. Let τ denote a generic irreducible cuspidal representation of $GSp_6(\mathbf{A})$. Denote by $L^S(\tau, Spin_7, s)$ the eight dimensional partial Spin L function attached to τ. It follows from [B-G], [V] and [G-J] that if this L function has a simple pole at $s = 1$, then the representation τ is a lift from a generic cuspidal representation π of the exceptional group $G_2(\mathbf{A})$. Let $E_\tau(g, s)$ denote the Eisenstein series defined on F_4 which is associated with the induce representation $Ind_{Q(\mathbf{A})}^{F_4(\mathbf{A})}\tau\delta_Q^s$. Here $Q = P_{\alpha_2,\alpha_3,\alpha_4}$ is the maximal parabolic subgroup of F_4 whose Levi part is GSp_6. The poles of this Eisenstein series are determined by $L^S(\tau, Spin_7, 8s-4)L^S(\tau, St, 16s-8)$. It follows from the assumption of τ, that the Eisenstein series has a simple pole at $s = 5/8$. Let $\mathcal{E}_\tau$ denote the residue representation at that point.

To construct a second example, let τ denote an irreducible cuspidal representation of $GL_2(\mathbf{A})$, and let π denote an irreducible cuspidal representation of $GL_3(\mathbf{A})$. Let $E_{\tau,\pi}(g, s)$ denote the Eisenstein series of F_4 associated with the induced representation $Ind_{R(\mathbf{A})}^{F_4(\mathbf{A})}(\tau\times\pi)\delta_R^s$. Here R is the maximal parabolic subgroup of F_4 whose Levi part contains the group $SL_2 \times SL_3$ generated by $\{x_{\pm(1000)}(r); x_{\pm(0010)}(r); x_{\pm(0001)}(r)\}$. The poles of this Eisenstein series are determined by

$$L^S(\tau\times\pi, 5(s-1/2))L^S(\mathrm{Sym}^2\tau\times\pi, 10s-5)L^S(\tau, 15(s-1/2))L^S(\pi, 20s-10)$$

Assume that π is the symmetric square lift of τ. Then the degree nine partial L function $L^S(\mathrm{Sym}^2\tau\times\pi, 10s-5)$ has a simple pole at $s = 3/5$. If also $L^S(\tau\times\pi, 1/2)$ is not zero, then the Eisenstein series $E_{\tau,\pi}(g, s)$ has a simple pole at $s = 3/5$, and we shall denote by $\mathcal{E}_{\tau,\pi}$ the residual representation at that point. We prove

Proposition 29. *With the above notations, we have* $\mathcal{O}(\mathcal{E}_\tau) = C_3$, *and* $\mathcal{O}(\mathcal{E}_{\tau,\pi}) = F_4(a_3)$.

Proof. We start with the representation $\mathcal{E}_\tau$. We need to prove two things. First we need to prove that $\mathcal{E}_\tau$, has no nonzero Fourier coefficients which corresponds to the unipotent orbits which are greater than the unipotent orbit C_3 or not related to it. It follows from [C] that we need to prove that $\mathcal{E}_\tau$, has no nonzero Fourier coefficients which corresponds to the unipotent orbits $B_3, F_4(a_2), F_4(a_1)$ and F_4. This we prove by a local argument. Indeed, let ν be a finite place such that the local constituent of $\mathcal{E}_\tau$, which we denote by $(\mathcal{E}_\tau)_\nu$, is unramified. Thus $(\mathcal{E}_\tau)_\nu = Ind_B^{F_4}\chi\delta_P^{1/8}\delta_B^{1/2}$. Here B is the standard Borel subgroup of F_4, and χ is an unramified character of B. We omit the reference to ν in the notations. Let T be the maximal torus of F_4, and we parameterize it as $h(t_1, t_2, t_3, t_4)$. Assume that $\chi(h(t_1, t_2, t_3, t_4) = \prod \chi_i(t_i)$ where χ_i are unramified characters. We assume that τ is a lift from the exceptional group G_2. Thus, the eight parameters of the Spin representation are $\chi_2\chi_3(p), \chi_2(p), \chi_3(p), 1, 1, \chi_3^{-1}(p), \chi_2^{-1}(p), \chi_2^{-1}(p)\chi_3^{-1}(p)$ where p is a generator of the maximal ideal in the ring of integers of F_ν. From this we obtain the two relations $\chi_1\chi_2\chi_3 = \chi_1\chi_2\chi_3\chi_4 = 1$. Let $w_0 = w[1213423]$. Then

$$(\chi\delta_P^{1/8})^{w_0}(h(t_1, t_2, t_3, t_4)) = (\chi\delta_P^{1/8})(h(t_1t_2t_3^{-2}, t_1t_2^2t_3^{-4}t_4^2, t_1t_2t_3^{-2}t_4, t_2t_3^{-1})) =$$

$$= \chi_1^{-2}\chi_2^{-4}\chi_3^{-2}\chi_4^{-1}(t_3)\chi_2^2\chi_3(t_4)|t_1t_2t_3^{-2}| = (\mu_\chi\delta_{B_3}^{1/2})(h(t_1, t_2, t_3, t_4))$$

Here $\mu_\chi(h(t_1, t_2, t_3, t_4)) = \chi_1^{-2}\chi_2^{-4}\chi_3^{-2}\chi_4^{-1}(t_3)\chi_2^2\chi_3(t_4)$ and B_3 is the Borel subgroup of GL_3 which contains the copy of SL_3 generated by $\{x_{\pm(1000)}(r); x_{\pm(0100)}(r)\}$. Hence, $Ind_B^{F_4}\chi\delta_P^{1/8}\delta_B^{1/2}$ which is isomorphic to $Ind_B^{F_4}(\chi\delta_P^{1/8})^{w_0}\delta_B^{1/2} = Ind_B^{F_4}\mu_\chi\delta_{B_3}\delta_L^{1/2}$ where L is the parabolic subgroup of F_4 whose Levi part is generated by T and $SL_3 =< x_{\pm(1000)}(r), x_{\pm(0100)}(r) >$. From this we conclude that $Ind_L^{F_4}\mu_\chi\delta_L^{1/2}$ is a constituent of $Ind_B^{F_4}\chi\delta_P^{1/8}\delta_B^{1/2}$ where now we view μ_χ as a character of L.

We now proceed as in [G-R-S5]. To prove that $\mathcal{E}_\tau$ has no nonzero Fourier coefficient with respect to a certain unipotent orbit, it is enough to show that $(\mathcal{E}_\tau)_\nu$ has no nonzero local functional which share the same invariant properties as the Fourier coefficient. From the above discussion, this corresponds to showing that $Ind_L^{F_4}\mu_\chi\delta_L^{1/2}$ has no embedding inside $Ind_V^{F_4}\psi_V$, where V is the unipotent group, and ψ_V is the character, which are associated with the unipotent orbit in question. For example, if $\mathcal{O} = F_4$, this corresponds to the case where V is the maximal unipotent subgroup of F_4, and ψ_V is the Whittaker character. Since $Ind_L^{F_4}\mu_\chi\delta_L^{1/2}$ has no nonzero Whittaker character, it follows that $(\mathcal{E}_\tau)_\nu$ has no nonzero corresponding functional, and hence $\mathcal{E}_\tau$ has no nonzero Fourier coefficient with respect to the unipotent orbit F_4. Next we consider the unipotent orbit B_3. The Fourier coefficients corresponding to this orbit are described right after Proposition 23. Thus, to prove the corresponding local result, it follows from Mackey theory that it is enough to prove the following. Given an element g in the space $L\backslash F_4/V$, there is an unipotent subgroup $\{x_\alpha(r)\}$ contained

in V such that $\psi_V(x_\alpha(r)) \neq 1$ and $gx_\alpha(r)g^{-1} \in L$. It follows from the definition of ψ_V as given before (104), that it is not trivial on $\{x_\alpha(r)\}$ where $\alpha \in \{(1000);(0100);(0120);(0122)\}$. Let w be an element in $L\backslash F_4/SL_3V$ where $SL_3 =< x_{\pm(0010)(r)}; x_{\pm(0001)}(r) >$. Then w can be chosen as a Weyl element. Thus, every representative of $L\backslash F_4/V$ can be written as wh where $w \in L\backslash F_4/V$, and $h \in SL_3$. If $w(1000) > 0$, then choosing $\alpha = (1000)$ we obtain $whx_{1000}(r)(wh)^{-1} \in L$. This follows from the fact that $\{x_{1000}(r)\}$ commutes with the above copy of SL_3. This eliminates most representatives in $L\backslash F_4/SL_3V$, and we are left with the following nine Weyl elements:

$$w[321];\ w[4321];\ w[324321];\ w[3214321];\ w[321324321];\ w[4321324321];$$

$$w[324321324321];\ w[3214321324321];\ w[321324321324321]$$

Thus we need to consider elements of the form wh where w is one of the above nine Weyl elements, and $h \in SL_3$. We have $wbw^{-1} \in L$ for w as above and B is the Borel subgroup of SL_3. Also, as follows from the description of the orbit B_3 right after Proposition 23, the group SO_3 embedded in SL_3 stabilizes the character ψ_V. Thus we may take $h \in B\backslash SL_3/SO_3$. Representatives of this space of double cosets are

$$A = \{e;\ w[3];\ w[4];\ w[34]x_{0011}(r);\ w[43]x_{0011}(r);\ w[434]x_{0001}(r_1)x_{0011}(r_2)\}$$

Going over all above nine Weyl elements w and all possible elements in the set A we can find a root α such that $\psi_V(x_\alpha(r)) \neq 1$ and that $(wa)x_\alpha(r)(wa)^{-1} \in L$ for all $a \in A$. For example, for the Weyl element $w[321324321]$, the root (0122) is suitable for all $a \in A$. Thus we deduce that $\mathcal{E}_\tau$ has no nonzero Fourier coefficient with respect to the unipotent orbit B_3. The other two orbits left are $F_4(a_1)$ and $F_4(a_2)$ are done in a similar way, and we shall omit the details.

Next we prove that $\mathcal{E}_\tau$ has a nonzero Fourier coefficient which is associated to the unipotent orbit C_3. In Section 2 this Fourier coefficient was described. We recall it now. Let V denote the unipotent subgroup of F_4 generated by all $\{x_\alpha(r)\}$ where we exclude the roots (1000); (0100) and (0010). Then the Fourier coefficient associated with the unipotent orbit C_3 is given by integral (73) where ψ_V is as follows. Write $v \in V$ as $v = x_{1110}(r_1)x_{0120}(r_2)x_{0001}(r_3)v'$. Then $\psi_V(v) = \psi(r_1 + r_2 + r_3)$. We shall assume that integral (73) is zero for all choice of data, and derive a contradiction. This assumption implies that the integral

$$\int\limits_{F\backslash\mathbf{A}}\ \int\limits_{V(F)\backslash V(\mathbf{A})} E(x_{0100}(m)v)\psi_V(v)dvdm$$

is zero for all choice of data. Let $w_0 = w[1234231]$. Then $E(w_0h) = E(h)$ for all $E \in \mathcal{E}_\tau$. Thus, we obtain that the integral

$$\int_{L_1(F)\backslash L_1(\mathbf{A})} \int_{V_1(F)\backslash V_1(\mathbf{A})} \int_{U_1(F)\backslash U_1(\mathbf{A})} E(u_1v_1l_1w_0)\psi_{U_1}(u_1)du_1dv_1dl_1 \tag{120}$$

is zero for all choice of data. Here U_1 is the maximal unipotent subgroup of Sp_6 embedded inside F_4. The character ψ_{U_1} is the Whittaker character of U_1. The unipotent group V_1 is generated by all $\{x_\alpha(r)\}$ where α is in the set

$$\{(1122);\ (1221);\ (1222);\ (1231);\ (1232);\ (1242);\ (1342);\ (2342)\}$$

The unipotent group L_1 is generated by all $\{x_{-\alpha}(r)\}$ where α is in the set

$$\{(1000);\ (1100);\ (1110);\ (1111);\ (1120)\}$$

In the following computations we will use the process of roots exchange. See subsection 2.2.2 for details. Expand integral (120) along the unipotent group $x_{1220}(m)$. For all $\gamma \in F$ we have by the left invariant property of E, that $E(x_{-1120}(\gamma)h) = E(h)$. Arguing as in (93) and (94), we collapse summation with integration, and deduce that integral (120) is equal to

$$\int_{\mathbf{A}} \int_{L_2(F)\backslash L_2(\mathbf{A})} \int_{V_2(F)\backslash V_2(\mathbf{A})} \int_{U_1(F)\backslash U_1(\mathbf{A})} E(u_1v_2l_2x_{-1120}(m)w_0)\psi_{U_1}(u_1)du_1dv_2dl_2dm \tag{121}$$

Here V_2 is the unipotent group generated by V_1 and $\{x_{1220}(r)\}$, and L_2 is the subgroup of L_1 generated by all roots in V excluding the root $-(1120)$. Next we expand integral (121) along the unipotent group $\{x_{1121}(m_1)x_{1120}(m_2)\}$. Using the group $\{x_{-(1111)}(r_1)x_{-(1110)}(r_2)\}$, integral (121) is equal to

$$\int_{\mathbf{A}^3} \int_{L_3(F)\backslash L_3(\mathbf{A})} \int_{V_3(F)\backslash V_3(\mathbf{A})} \int_{U_1(F)\backslash U_1(\mathbf{A})} E(u_1v_3l_3z(m_1,m_2,m_3)w_0)\psi_{U_1}(u_1)du_1dv_3dl_3dm_j \tag{122}$$

Here $z(m_1,m_2,m_3) = x_{-(1120)}(m_1)x_{-(1111)}(m_2)x_{-(1110)}(m_3)$ and the group V_3 is generated by V_2 and $\{x_{1121}(r), x_{1120}(r)\}$. The group L_3 is generated by all $\{x_{-\alpha}(r)\}$ where α is in the set of roots $\{(1000);\ (1100);\ (1110)\}$. Arguing as in [Ga-S] we deduce that integral (122) is zero for all choice of data if and only if the inner integration over the group U_1, V_3 and L_3 is zero for all choice of data. Next we expand the inner integration along the unipotent group $\{x_{1111}(r)\}$. The contribution from the nontrivial orbit is zero. Indeed, this contribution produces a Fourier coefficient which is associated to the unipotent orbit $F_4(a_2)$. By the first part of the Proposition, the representation $\mathcal{E}_\tau$ do not have a nonzero Fourier coefficients corresponding to this unipotent orbit. Hence we are left with the contribution of the constant term. As in the expansions in integrals (121) and (122) we expand along $\{x_{1110}(r)\}$ and use for it

the group $\{x_{-(1100)}(r)\}$. Then we repeat the same process with $\{x_{1100}(r)\}$ and $\{x_{-(1000)}(r)\}$. Hence, the integral

$$\int\limits_{V_4(F)\backslash V_4(\mathbf{A})}\int\limits_{U_1(F)\backslash U_1(\mathbf{A})} E(u_1v_4)\psi_{U_1}(u_1)du_1dv_4 \tag{123}$$

is zero for all choice of data. Here V_4 is the unipotent subgroup of $U(C_3)$ generated by all $x_\alpha(r) \in U(C_3)$ excluding the root (1000). Here $U(C_3) = P_{\alpha_2,\alpha_3,\alpha_4}$. Finally, we expand integral (123) along the unipotent group x_{1000}. The nontrivial orbit contributes zero, since the Fourier coefficient obtained is associated with the unipotent orbit F_4. Thus we are left only with the constant term. From this we deduce that the integral

$$\int\limits_{U_1(F)\backslash U_1(\mathbf{A})} E^{U(C_3)}(u_1)\psi_{U_1}(u_1)du_1 \tag{124}$$

is zero for all choice of data. However, from the definition of $\mathcal{E}_\tau$ and from the fact that τ is generic this is a contradiction. This concludes the proof of the Proposition for the representation $\mathcal{E}_\tau$.

Next we consider the representation $\mathcal{E}_{\tau,\pi}$. For simplicity we shall assume that τ has a trivial central character. Since we assume that $L^S(\mathrm{Sym}^2\tau \times \pi, 10s-5)$ has a simple pole at $s = 3/5$, this means that π is the symmetric square lift of τ. Thus, if $(\mathcal{E}_{\tau,\pi})_\nu$ is the unramified constituent of $\mathcal{E}_{\tau,\pi}$ at a finite place ν, then it is isomorphic to $Ind_B^{F_4}\bar{\chi}\delta_R^{1/10}\delta_B^{1/2}$. Here $\bar{\chi}$ is the character of T given by $\bar{\chi}(h(t_1,t_2,t_3,t_4)) = \chi^2(t_1t_3t_4)\chi^{-3}(t_2)$ which is extended trivially to B. Let $w_0 = w[2132134324]$. Then

$$(\bar{\chi}\delta_R^{1/10})^{w_0}(h(t_1,t_2,t_3,t_4)) = (\bar{\chi}\delta_R^{1/10})(h(t_1t_2^{-1}t_4^2, t_1^2t_2^{-3}t_3^2t_4^2, t_1t_2^{-2}t_3^2t_4, t_1t_2^{-1}t_3)) =$$

$$\chi(t_2)|t_1^2t_2^{-3}t_3^2t_4^2|^{1/2} = \mu_\chi\delta_{B_2\times B_3}^{1/2}(h(t_1,t_2,t_3,t_4)$$

Here $\mu_\chi(h(t_1,t_2,t_3,t_4)) = \chi(t_2)$ and $B_2 \times B_3$ is the Borel subgroup of the Levi part of the maximal parabolic subgroup R. Arguing as in the previous case, we deduce that $(\mathcal{E}_{\tau,\pi})_\nu$ is the unramified constituent of $Ind_R^{F_4}\mu_\chi\delta_R^{1/2}$.

To prove that $\mathcal{O}(\mathcal{E}_{\tau,\pi}) = F_4(a_3)$ we first need to prove that $\mathcal{E}_{\tau,\pi}$ has no nonzero Fourier coefficient associated with any unipotent orbit which is greater than $F_4(a_3)$. This is done by showing that the local constituent $(\mathcal{E}_{\tau,\pi})_\nu$ at an unramified finite place cannot support a suitable functionals. This is done by a double coset argument in the same way as for the representation $\mathcal{E}_\tau$, and hence will be omitted.

To complete the proof we need to show that $\mathcal{E}_{\tau,\pi}$ has a nonzero Fourier coefficient associated with the unipotent orbit $F_4(a_3)$. We first show that it has a nonzero Fourier coefficient associated with the unipotent orbit $\widetilde{A}_2 + A_1$. To prove that we need to show that integral (73) is not zero for some choice of data. Here V is the unipotent group defined as follows. Let

$V' = U_{\alpha_1,\alpha_3}$. Its dimension is 22. Let V be the subgroup of V' generated by all $x_\alpha(r) \in V'$ excluding the roots

$$\{(0100);\ (1100);\ (0110);\ (1110);\ (0120);\ (1120);\ (0001);\ (0011)\}$$

The character ψ_V is defined as follows. Write $v \in V$ as $v = x_{0121}(r_1)x_{1111}(r_2)x_{1220}(r_3)v'$. Then $\psi_V(v) = \psi(r_1 + r_2 + r_3)$. We shall assume that integral (73) is zero for all choice of data, and derive a contradiction. Let $w_0 = w[213213432]$. Using the left invariance property of E, we deduce that the integral

$$\int\limits_{L_1(F)\backslash L_1(\mathbf{A})} \int\limits_{V_1(F)\backslash V_1(\mathbf{A})} \int\limits_{U_1(F)\backslash U_1(\mathbf{A})} E(u_1v_1l_1w_0)\psi_{U_1}(u_1)du_1dv_1dl_1 \tag{125}$$

is zero for all choice of data. Here U_1 is the maximal unipotent subgroup of $SL_2 \times SL_3$ which is contained in the Levi part of R. The character ψ_{U_1} is the Whittaker character of this group. The group V_1 is generated by all $\{x_\alpha(r)\}$ where α is a root in the set $\{(1242);\ (1232);\ (1122);\ (1121);\ (0122)\}$. The group L_1 is generated by all $\{x_{-\alpha}(r)\}$ where α is a root in the set $\{(1221);\ (1220);\ (1100);\ (0110);\ (0100)\}$. Since integral (125) is zero for all choice of data, then any of its Fourier coefficients is zero. Thus, we deduce that the integral

(126)
$$\int\limits_{(F\backslash A)^2} \int\limits_{L_1(F)\backslash L_1(\mathbf{A})} \int\limits_{V_1(F)\backslash V_1(\mathbf{A})} \int\limits_{U_1(F)\backslash U_1(\mathbf{A})} E(u_1x_{1342}(r_1)x_{2342}(r_2)v_1l_1w_0)\psi_{U_1}(u_1)du_1dv_1dl_1dr_1dr_2$$

is zero for all choice of data. Next we expand integral (126) along the unipotent group $\{x_{1231}(r)\}$. Using the unipotent group $\{x_{-(1221)}(r)\}$, and arguing in a similar way as in the integrals (93) and (94), we deduce that the integral

$$\int\limits_{\mathbf{A}} \int\limits_{L_2(F)\backslash L_2(\mathbf{A})} \int\limits_{V_2(F)\backslash V_2(\mathbf{A})} \int\limits_{U_1(F)\backslash U_1(\mathbf{A})} E(u_1v_2l_2x_{-(1221)}(m)w_0)\psi_{U_1}(u_1)du_1dv_2dl_2dm$$

is zero for all choice of data. Here V_2 is the group generated by V_1 and $\{x_\alpha(r)\}$ where α is in the set $\{(1231);\ (1342);\ (2342)\}$. The group L_2 is the subgroup of L_1 excluding $\{x_{-(1221)}(r)\}$. We can continue this process. The vanishing assumption implies either that any Fourier coefficient of the integral is zero, or we can perform, as above, Fourier expansions and use collapsing of summation with integration as in a similar way as in (93) and (94). Eventually, we deduce that the integral

$$\int\limits_{L_1(\mathbf{A})} \int\limits_{U_1(F)\backslash U_1(\mathbf{A})} E^{U(R)}(u_1l_1)\psi_{U_1}(u_1)du_1dl_1 \tag{127}$$

is zero for all choice of data. Here $U(R)$ is the unipotent radical of R. Arguing as in [Ga-S] we may deduce that the inner integration of integral (127) is zero for all choice of data. However, from the definition of $\mathcal{E}_{\tau,\pi}$ this is not so. Hence we derived a contradiction.

From this we deduce that $\mathcal{O}(\mathcal{E}_{\tau,\pi})$ is at least $\widetilde{A}_2 + A_1$. In fact, we claim that $\mathcal{O}(\mathcal{E}_{\tau,\pi})$ cannot be equal to $\widetilde{A}_2 + A_1$. Indeed, suppose that there is an equality. The stabilizer of the unipotent orbit $\widetilde{A}_2 + A_1$ is a group of type A_1. If we consider integral (74) which corresponds to the unipotent orbit $\widetilde{A}_2 + A_1$, it follows that the function $f(g)$ defines an automorphic function of $\widetilde{SL}_2(\mathbf{A})$. Hence, for some $\beta \in F^*$, the integral

$$\int\limits_{F\backslash \mathbf{A}} f\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \psi(\beta x) dx$$

is not zero for some choice of data. This nonzero integral is a Fourier coefficient which corresponds to a unipotent orbit which is greater than $\widetilde{A}_2 + A_1$.

Hence $\mathcal{O}(\mathcal{E}_{\tau,\pi}) > \widetilde{A}_2 + A_1$ and $\mathcal{O}(\mathcal{E}_{\tau,\pi}) \geq C_3(a_1)$. The stabilizer of the orbit $C_3(a_1)$ contains a split group of type A_1. Arguing in a similar way as above, we deduce that $\mathcal{O}(\mathcal{E}_{\tau,\pi}) > C_3(a_1)$, or that $\mathcal{O}(\mathcal{E}_{\tau,\pi}) \geq F_4(a_3)$. But from the local argument introduced at the beginning of the proof, we know that $\mathcal{O}(\mathcal{E}_{\tau,\pi}) \leq F_4(a_3)$. Hence we get $\mathcal{O}(\mathcal{E}_{\tau,\pi}) = F_4(a_3)$.

□

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